Assignment 2: Due Wednesday November 18

Math 270A: Techniques in Scientific Computing

1. The 1D heat equation:

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1), \quad t \in [0,T], \quad \text{periodic boundary conditions}$$

$$u(x,0) = \begin{cases} 0 & x \in [0,.25] \cup [.75,1] \\ 1 & x \in [.25,.75]. \end{cases}$$

a. Write a C++ program for Forward Euler:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mu \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta x^2}, \quad i = 0,1,...,m-1 \quad n = 0,1,...,N$$

Use $T = 5$, $\mu = 1$ and periodic boundary conditions. Run with $\Delta x = \frac{1}{m}$ with $m = 32, 64, 128, 256, 512$ and $N = 10m^2$, $\Delta t = \frac{T}{N}$ i.e. $\Delta t = .5\Delta x^2$. Use Matlab (or something like that) to plot your solution at $t^n = 0,1,2,3,4,5$. This should take a long time when Δx is small (i.e. Δt proportionate to Δx^2 is unacceptably slow). What happens if you attempt to take large time steps and keep $N = 150$?

b. Use your multigrid solver to do Crank-Nicholson instead:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mu \left(\frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{2\Delta x^2} + \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{2\Delta x^2} \right), \quad i = 0,1,...,m-1 \quad n = 0,1,...,N$$

Use $T = 5$, $\mu = 1$ and periodic boundary conditions. Run with $\Delta x = \frac{1}{m}$ with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Plot your solution at $t^n = 0,1,2,3,4,5$. This should be much faster than Forward Euler.

2. The 2D heat equation:

$$\frac{\partial u}{\partial t} = \mu \Delta u, \quad \mathbf{x} = (x,y) \in [0,1] \times [0,1], \quad t \in [0,T], \quad \text{periodic boundary conditions}$$

$$u(x,0) = \begin{cases} 0 & x \notin [.25,.75] \times [.25,.75] \\ 1 & x \in [.25,.75] \times [.25,.75]. \end{cases}$$

Use your multigrid solver to do Crank-Nicholson:

$$\frac{u_{(i,j)}^{n+1} - u_{(i,j)}^n}{\Delta t} = \mu \left(\frac{u_{(i+1,j)}^{n+1} + u_{(i-1,j)}^{n+1} - 2u_{(i,j)}^{n+1}}{2\Delta x^2} + \frac{u_{(i+1,j)}^n + u_{(i-1,j)}^n - 2u_{(i,j)}^n}{2\Delta x^2} \right) +$$

$$\mu \left(\frac{u_{(i,j+1)}^{n+1} + u_{(i,j-1)}^{n+1} - 2u_{(i,j)}^{n+1}}{2\Delta y^2} + \frac{u_{(i,j+1)}^n + u_{(i,j-1)}^n - 2u_{(i,j)}^n}{2\Delta y^2} \right)$$

$$i,j = 0,1,...,m-1 \quad n = 0,1,...,N$$

Use $T = 5$, $\mu = 1$ and periodic boundary conditions. Run with $\Delta x = \Delta y = \frac{1}{m}$ with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Use Matlab (or something like that) to plot your solution at $t^n = 0,1,2,3,4,5$.

3. The 1D wave equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x \in [0,1], \quad t \in [0,T], \quad \text{periodic boundary conditions}$$

$$u(x,0) = \begin{cases} 0 & x \in [0,.25] \cup [.75,1] \\ 1 & x \in [.25,.75]. \end{cases}$$

a. Use $a = .25$, $T = 5$ and upwinding advection:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Use $m = 32, 64, 128, 256, 512$ and $N = \frac{mT}{4}$ (i.e. $\Delta t = \frac{1}{2m}$). Plot the exact solution and your numerical solution (on the same graph) at $t^n = 0, 1, 2, 3, 4, 5$. What happens if you try and take large time steps by keeping $N = 150$ (i.e. $\Delta t = \frac{1}{30}$)?

b. Use $a = .25$, $T = 5$ and semi-Lagrangian advection with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Plot the exact solution and your numerical solution (on the same graph) at $t^n = 0, 1, 2, 3, 4, 5$.

c. Use $a = .25$, $T = 5$ and unconditionally stable McCormack advection with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Plot the exact solution and your solution (on the same graph) at $t^n = 0, 1, 2, 3, 4, 5$.

4. The 1D advection diffusion equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial^2 x}, \quad x \in [0,1], \quad t \in [0,T], \quad \text{periodic boundary conditions}$$

$$u(x,0) = \begin{cases} 0 & x \in [0,.25] \cup [.75,1] \\ 1 & x \in [.25,.75]. \end{cases}$$

a. Use $a = .25$, $\mu = 1$, $T = 5$ and upwinding advection/implicit diffusion:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = \mu\frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{\Delta x^2}$$

Use $m = 32, 64, 128, 256, 512$ and $N = \frac{mT}{4}$ (i.e. $\Delta t = \frac{1}{2m}$). Plot the exact solution and your numerical solution (on the same graph) at $t^n = 0, 1, 2, 3, 4, 5$. What happens if you try and take large time steps by keeping $N = 150$ (i.e. $\Delta t = \frac{1}{30}$)?

b. Use $a = .25$, $\mu = 1$, $T = 5$ and semi-Lagrangian advection with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Plot the exact solution and your numerical solution (on the same graph) at $t^n = 0, 1, 2, 3, 4, 5$.

c. Use $a = .25$, $\mu = 1$, $T = 5$ and unconditionally stable McCormack advection with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Plot the exact solution and your solution (on the same graph) at $t^n = 0, 1, 2, 3, 4, 5$.

5. The 2D wave equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \mathbf{x}}\mathbf{a} = 0, \quad \mathbf{x} = (x,y) \in [0,1] \times [0,1], \quad t \in [0,T], \quad \text{periodic boundary conditions}$$

$$u(\mathbf{x},0) = \left\{ \begin{array}{ll} 0 & \mathbf{x} \notin [.25,.75] \times [.25,.75] \\ 1 & \mathbf{x} \in [.25,.75] \times [.25,.75]. \end{array} \right.$$

a. Use $\mathbf{a} = (.25, .1)$, $T = 5$ and semi-Lagrangian advection with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Plot the exact solution and your solution at $t^n = 0, 1, 2, 3, 4, 5$.

b. Use $\mathbf{a} = (.25, .1)$, $T = 5$ and unconditionally stable McCormack advection with $m = 32, 64, 128, 256, 512$ and $N = 150$ (i.e. $\Delta t = \frac{1}{30}$). Plot your solution at $t^n = 0, 1, 2, 3, 4, 5$.

6. Write a C++ program to solve the Stokes equations:

$$\begin{array}{l} -\mu\Delta\mathbf{u}^* = \mathbf{f} \\ \Delta\hat{p} = -\nabla\cdot\mathbf{u}^* \\ \mathbf{u} = \mathbf{u}^* + \nabla\hat{p} \\ p = \mu\Delta\hat{p} \end{array}, \quad \mathbf{x} = (x,y) \in [0,1]\times[0,1], \quad t \in [0,T] \text{ periodic boundary conditions.}$$

Here, set $\mu = 1$ and $T = 5$. Use your multigrid solver and a MAC grid to solve the 3 Poisson equations and to compute $\nabla\cdot\mathbf{u}^*$. Use

$$\mathbf{f}(\mathbf{x},t) = \left(\begin{array}{c} -8\pi^2 \mathrm{e}^{\sin(\frac{2\pi t}{5})}\sin(2\pi x)\cos(2\pi y) + 16x(1-x)(1-2x) \\ 8\pi^2 \mathrm{e}^{\sin(\frac{2\pi t}{5})}\cos(2\pi x)\sin(2\pi y) + 16y(1-y)(1-2y) \end{array} \right).$$

The exact solution should be

$$\mathbf{u}(\mathbf{x},t) = \left(\begin{array}{c} -\mathrm{e}^{\sin(\frac{2\pi t}{5})}\sin(2\pi x)\cos(2\pi y) \\ \mathrm{e}^{\sin(\frac{2\pi t}{5})}\cos(2\pi x)\sin(2\pi y) \end{array} \right)$$

and $p(\mathbf{x}) = 8\left[x(1-x)\right]^2 + 8\left[y(1-y)\right]^2$. Visualize your solution by comparing marker particle trajectories under your numerical solution and under the exact solution at $t^n = 0, 1, 2, 3, 4, 5$. Use $m = 512$, $\Delta x = \Delta y = \frac{1}{512}$ and $\Delta t = \frac{1}{30}$.

7. Write a C++ program to solve the Navier-Stokes equations:

$$\begin{array}{l} \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\mathbf{u}}{\partial\mathbf{x}}\mathbf{u}\right) = \mu\Delta\mathbf{u} + \nabla p + \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(\mathbf{x},0) = \mathbf{0} \end{array}, \mathbf{x} \in [0,1]\times[0,1], \quad t \in [0,T], \text{ periodic boundary conditions}$$

Set $\mu = 1$, $\rho = 1$. Use the splitting discussed in class:

$$\begin{array}{l} \rho\left(\frac{\mathbf{u}^*-\mathbf{u}^n}{\Delta t} + \frac{\partial\mathbf{u}^n}{\partial\mathbf{x}}\mathbf{u}^n\right) = \mu\Delta\mathbf{u}^* + \mathbf{f}^{n+1} \\ \Delta p^{n+1} = -\frac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^* \\ \mathbf{u}^{n+1} = \mathbf{u}^* + \frac{\Delta t}{\rho}\nabla p^{n+1} \end{array}$$

Use the same source $\mathbf{f}$ as in the Stokes flow problem. Visualize your solution by plotting marker particle trajectories under your numerical solution (I don't know the analytic solution here so you can't compare with that this time) $t^n = 0, 1, 2, 3, 4, 5$. Use $m = 512$, $\Delta x = \Delta y = \frac{1}{512}$ and $\Delta t = \frac{1}{30}$.

a. Use semi-Lagrangian advection and your muitigrid solver for the heat equations and Poisson equation.

b. Use second order McCormack advection and your multigrid solver for the heat equations and Poisson equation.

8. Repeat problem 7 but with $\mu = 1, .5, .25, .1$, $\mathbf{f} = \mathbf{0}$ and

$$\mathbf{u}(\mathbf{x},0) = \left(\begin{array}{c} \frac{\partial}{\partial y}\phi(\mathbf{x}) \\ -\frac{\partial}{\partial x}\phi(\mathbf{x}) \end{array} \right)$$

where

$$\phi(\mathbf{x}) = \begin{cases} 256\left(|\mathbf{x} - (.25, .25)|_2 - .25\right)^4, & \text{when} \quad |\mathbf{x} - (.25, .25)|_2 - .25 < 0 \\ 10000\left(|\mathbf{x} - (.75, .75)|_2 - .1\right)^4, & \text{when} \quad |\mathbf{x} - (.75, .75)|_2 - .1 < 0 \\ 0 \ \text{ otherwise.} \end{cases}$$

9. Show that the pressure in the splitting scheme for problem 7 minimizes the kinetic energy ($\frac{1}{2}\int_\Omega \rho\, |\mathbf{u}|_2^2\, d\mathbf{x}$) of the fluid at time t^{n+1}.