Assignment 1: Due Wednesday, October 21.

Math 270A: Techniques in Scientific Computing

1. Let $\mathbf{F} = \frac{\partial \phi}{\partial \mathbf{X}} : \Omega^0 \times [0,T] \to \mathbb{R}^{2\times 2}$ (i.e. $\mathbf{F}$ is the flow map Jacobian). Show that

$$\frac{\partial}{\partial t}\det(\mathbf{F}) = \det(\mathbf{F})\mathrm{tr}(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial t}).$$

2. Use the result in 1 to show that $\det(\mathbf{F}) = 1$ iff $\nabla \cdot \mathbf{u} = 0$. Here $\mathbf{u} : \Omega^t \times [0,T] \to \mathbb{R}^2$ is the Eulerian velocity $\mathbf{u}(\mathbf{x},t) = \frac{\partial \phi}{\partial t}(\phi^{-1}(\mathbf{x},t),t)$.

3. Consider the projection form of the periodic Stokes equations:

$$\mu\Delta\mathbf{u}^* + \mathbf{f}^{ext} = \mathbf{0}$$
$$\Delta\hat{p} = \nabla \cdot \mathbf{u}^*$$
$$\mathbf{u} = \mathbf{u}^* - \nabla\hat{p}$$
$$p = \mu\Delta\hat{p}.$$

Show that equal and opposite forces $\mathbf{f}^{ext}$ produce equal and opposite flows. This fact shows the reversibility of a Stokes flow. That is, if we mix a Stokesian fluid, it can be unmixed exactly by just repeating the mixing in reverse.

4. Write a program in C++ to solve the 1D Poisson equation with periodic boundary conditions:

$$-\frac{\partial^2 u}{\partial x^2} = 4\pi^2\sin(2\pi x), \quad x \in [0,1].$$

Use second order centered differences and a multigrid solver (with residual tolerance of $1e{-}15$) for the linear system.

a. Plot your solution using matlab or python with grid resolutions of N=32,64,128,256 and 512.

b. Show that the L^∞ error is second order by plotting

$$\frac{\log(\left|\mathbf{e}^{\Delta x}\right|_\infty)}{\log(\Delta x)}, \quad \Delta x = \frac{1}{32}, \frac{1}{64}, \ldots \frac{1}{512}.$$

Here, $\left|\mathbf{e}^{\Delta x}\right|_\infty = \max_{i=1:\mathrm{N}} |e_i^{\Delta x}|$, $e_i^{\Delta x} = u_i^{\text{exact}} - v_i^{\Delta x}$, u_i^{exact} is the solution of the PDE at the i^{th} grid node and $v_i^{\Delta x}$ is the solution of the discretized problem with resolution Δx at the i^{th} grid node.

c. For $\Delta x = \frac{1}{512}$, plot the convergence rate of the multigrid method at each iteration. This should be very close to .1 asymptotically if your code is working properly.

5. Show that the multigrid V-cycle can be written as a linear operator on the right hand side when the initial guess is $\mathbf{0}$. That is show that the n^{th} iterate in the multigrid method is given as

$$\mathbf{x}^n = M^n\mathbf{b}$$

where $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$ where $\mathbf{A}\mathbf{x} = \mathbf{b}$ (i.e. applying a v-cycle update to your approximation is equivalent to applying M to your current approximation). Show that M is symmetric for one pass of pre and post Jacobi smoothing but that it is not necessarily symmetric with one pass of pre and post Gauss-Seidel smoothing.

6. Write a program in C++ to solve the 2D Poisson equation with periodic boundary conditions:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 8\pi^2 \sin(2\pi x)\cos(2\pi y), \quad x \in [0,1].$$

Use second order centered differences and a multigrid solver (with residual tolerance of $1e-15$) for the linear system.

a. Plot your solution using matlab or python with grid resolutions of 32,64,128,256 and 512.

b. Show that the L^∞ error is second order by plotting

$$\frac{\log(\left|\mathbf{e}^{\Delta x}\right|)}{\log(\Delta x)}, \quad \Delta x = \frac{1}{32}, \frac{1}{64}, \ldots \frac{1}{512}.$$

c. For $\Delta x = \Delta y = \frac{1}{512}$, plot the convergence rate of the multigrid method at each iteration.