1. Consider the Poisson interface problem

$$-\frac{\partial}{\partial x} (\beta \frac{\partial u}{\partial x}) = g, \ x \in (0, 1)/\Gamma, \ \Gamma \in (0, 1)$$  \hspace{1cm} (1)

with boundary conditions

$$u(0) = a, \ u(1) = b$$

and jump conditions

$$\beta^+ \frac{\partial u^+}{\partial x} (\Gamma) - \beta^- \frac{\partial u^-}{\partial x} (\Gamma) = c, \ u^+(\Gamma) - u^-(\Gamma) = d.$$

Here the coefficient is discontinuous with $\beta = \begin{cases} e, \ x < \Gamma \\ f, \ x > \Gamma \end{cases}$.

(a) Derive the weak form of this problem and show that it is equivalent to the above strong form under certain assumptions.

(b) Derive the Green’s function for this problem.

(c) Consider the two grids $x_i = i\Delta x, \ i = 0, 1, \ldots, N$ with $\Gamma \in (x_{N-1}, x_N)$ and $x_j = x_{N-1} + (j - N - 1)\Delta x, \ j = N + 1, N + 2, \ldots, M$ with $x_M = 1$. The grids overlap in the interval $(x_{N-1}, x_N) = (x_{N+1}, x_{N+1})$ which contains the interface point $\Gamma$. Derive the FEM discretization using piecewise linear interpolation over the two grids. These functions have degrees of freedom $u_i$ and define a function with a discontinuity at $\Gamma$ through

$$u^{\Delta x} = \begin{cases} \sum_{i=0}^N u_i N_i, & x \in (0, \Gamma) \\ \sum_{j=N+1}^M u_j N_j, & x \in (\Gamma, 1) \end{cases}$$  \hspace{1cm} (2)

where the $N_i, N_j$ are the piecewise linear interpolation basis functions over the respective grids.

(d) Show that the solution to the FEM discretization in Equation (2) from part (c) satisfies $u_i = u(x_i)$ where $u$ is the solution to the interface problem in Equation (1).

2. Consider the biharmonic equation

$$\frac{\partial^4 u}{\partial x^4} = f, \ x \in (0, 1)$$  \hspace{1cm} (3)

$$u(0) = a, \frac{\partial u}{\partial x}(0) = b$$

$$u(1) = c, \frac{\partial u}{\partial x}(1) = d$$

Show that the FEM solution (using piecewise cubic Hermite interpolation) $u^{\Delta x} = \sum_i u_i N_i$ satisfies $u^{\Delta x}(x_i) = u(x_i)$ where $x_i$ are the discrete grid locations in the FEM problem and $u$ is the solution to the biharmonic problem in Equation (3).
3. Consider the Poisson problem with Robin boundary conditions

\[-\frac{\partial^2 u}{\partial x^2} = f, \ x \in (0, 1)\]

\[au(0) + b \frac{\partial u}{\partial x}(0) = c, \ u(1) = d\]

where \(ab < 0\).

(a) Derive the weak form.

(b) Derive the Green’s function.

2 Programming

1. Use piecewise linear interpolation and FEM to discretize the Poisson problem

\[-\frac{\partial^2 u}{\partial x^2}(x) = \sin(x), \ x \in (0, 2\pi)\]

with boundary conditions \(u(0) = 0, u(2\pi) = 1\). Verify that you get the exact solution at the grid nodes if you integrate \(\int_0^1 N_i(x) \sin(x)dx\) exactly.

2. Implement the FEM discretization derived in Pen and Paper Problem 1(c). Verify that you get the exact solution to the interface Poisson problem when \(g = 0, a = 0, b = 1, c = 2, d = 3, e = 4\) and \(f = 5\).

3. Implement an FEM discretization derived from the weak from in Pen and Paper Problem 3 with \(a = 1, b = -1\) and \(f(x) = \cos(x)\). Should the FEM approximation be exact at the grid nodes?

4. Write a program that discretizes the the Poisson equation over the unit square domain (without holes) shown in Figure 1 below. Use \(u(x, y) = \sin(2\pi x) \cos(2\pi y)\) as your exact solution. Use Dirichlet boundary conditions over the left most wall of the domain and appropriate Neumann conditions over the remaining portion of the boundary.

5. Write a problem that discretizes the linear elasticity problem over the unit square domain (without holes) shown in Figure 1 below. Use \(\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\) and \(\mu = \frac{E}{2(1+\nu)}\) with \(E = 100\) and \(\nu = 3\). Plot your solution as a deformed version of the mesh. Use zero Dirichlet (displacement) boundary conditions over the left most wall of the domain and zero Neumann over the remaining portion of the boundary. Use body force \(f = (0, -9.8)^T\).
Figure 1: **A mesh with holes.** This can be loaded into matlab with the commands `load 'mesh_with_holes.dat'` and `load 'nodes.dat'`. The first command creates an array (which will be a variable called mesh with holes after the call) with mesh connectivity information (with indexing starting from 1 not 0) and the second creates an array (called nodes) that lists the vertices in the mesh.