Homework 1, Due Friday, April 25
Math 269C: Numerical Methods for Elliptic Equations

I recommend using MATLAB or Python for solving the linear systems in the programming problems, but you are welcome to use whatever platform you prefer.

1. Consider the interface problem

\[-u_{xx} = f, \quad x \in (0, 1) \setminus \{\hat{x}\}, \quad \hat{x} \in (0, 1)\]

\[\left[u\right]_{\hat{x}} = a\]

\[\left[u_x\right]_{\hat{x}} = b\]

\[u(0) = u_D, \quad u_x(1) = g\]

In this problem, the solution \(u\) can have discontinuities at \(\hat{x}\). Here, \(\left[u\right]_{\hat{x}} = \lim_{x \to \hat{x}^+} u(x) - \lim_{x \to \hat{x}^-} u(x)\). Derive the weak and energy forms of this problem.

2. Consider the Neumann problem

\[-u_{xx} = f, \quad x \in (0, 1)\]

\[u(0) = u_D, \quad u_x(1) = g\]

Implement the finite difference discretization:

\[\Delta x = \frac{1}{N-1}, \quad v_0 = u_D, \quad \frac{-v_{i-1} - v_{i+1} + 2v_i}{\Delta x^2} = f_i, \quad i = 1, \ldots, N-2, \quad \frac{v_{N-1} - v_{N-2}}{\Delta x} = \frac{g}{\Delta x}\]

Use \(f(x) = 4\pi^2\sin(2\pi x), \quad u_D = 0\) and \(g = 2\pi\). In this case \(u(x) = \sin(2\pi x)\) is the exact solution. The error in the approximate solution at grid node \(x_i = i\Delta x\) is then

\[e_i = v_i - \sin(2\pi x_i)\]

If \(e \in \mathbb{R}^N\) denotes the vector whose entries indicate the error at each grid node, denote its \(\infty\)-norm as

\[|e|_\infty = \max |e_i|\]

We would expect that \(|e|_\infty \approx C\Delta x^p\) for some \(C > 0\) and some \(p < 0\) (the smaller \(p\) and \(C\) the better). We can empirically estimate them by examining \(\log(|e|_\infty) \approx C + p\log(\Delta x)\). Do this by plotting \(\log(|e|_\infty)\) versus \(\log(\Delta x)\) for a few different values of \(\Delta x\) and then finding the best fit line to the data. Specifically, use \(N = 33, 65, 129\) and \(257\).

3. Repeat problem 2 with the finite volume scheme

\[\Delta x = \frac{1}{N - \frac{1}{2}}, \quad v_0 = u_D, \quad \frac{2v_i - v_{i-1} - v_{i+1}}{\Delta x} = f_i, \quad i = 1, 2, \ldots, N-2, \quad \frac{v_{N-1} - v_{N-2}}{\Delta x} = f_{N-1}\Delta x + g\]

4. Repeat problem 2 with the finite element scheme

\[\Delta x = \frac{1}{N - 1}, \quad v_0 = u_D, \quad \frac{2v_i - v_{i-1} - v_{i+1}}{\Delta x} = F_i, \quad i = 1, 2, \ldots, N-2, \quad \frac{v_{N-1} - v_{N-2}}{\Delta x} = F_{N-1} + g\]

where \(F_i = \int_0^1 4\pi^2\sin(2\pi x)N_i(x)\,dx\). Compute the integrals exactly.

5. Design a FEM discretization of the interface problem 1. Explain your motivation for your ideas. Test it with \(u(x) = \sin(2\pi x)\) for \(x < \hat{x}\) and \(u(x) = x\) for \(x > \hat{x}\). Use \(\hat{x} = 2/3\). You do not need to do an empirical error analysis.