Math 269B: Periodic grid functions and Fourier transform

1 Periodic grid functions

Let $\Delta x = \frac{2\pi}{N+1}$ and $x_i = i\Delta x$ for $i = 0, 1, \ldots, N$ and consider functions values $f_i$ defined at these grid nodes $x_i$. We call these periodic grid functions where we define $f_i = f_{\text{mod}(i,N+1)}$ for $i \in \mathbb{Z}$.

2 Trigonometric interpolation

Assuming $N$ is even, these functions can be written as

$$f_i = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} \hat{f}_\omega e^{i\omega x_i}$$

where the $\hat{f}_\omega$ are uniquely determined from the $f_i$. We can call $f(x) = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} \hat{f}_\omega e^{i\omega x}$ the trigonometric interpolant of the $f_i$ since $f(x_i) = f_i$. To show that we can always define this interpolant (and that it is unique), we need only show that the vectors $v_\omega = \begin{pmatrix} 1 & e^{i\omega x_1} & e^{i\omega x_2} & \cdots & e^{i\omega x_N} \end{pmatrix} \in \mathbb{C}^{N+1}$ where $\omega \in \{-N/2, \ldots, N/2\}$ form an orthogonal basis for $\mathbb{C}^{N+1}$. If we do this, the result follows because the expression

$$f_i = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} \hat{f}_\omega e^{i\omega x_i} \text{ for } i = 0, 1, \ldots, N$$

is just the expression of the vector $f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix} \in \mathbb{C}^{N+1}$ in the basis defined by the $v_\omega$.

We can show this from the following property of the trigonometric basis functions

$$(v_\nu, v_\mu)_{\Delta x} = \sum_{j=0}^{N} e^{i(\mu-\nu)x_j} \Delta x = \begin{cases} 2\pi, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$$

where

$$(f, g)_{\Delta x} = \sum_{j=0}^{N} f_j g_j \Delta x$$

for any $f, g \in \mathbb{C}^{N+1}$. We can see this from (assuming $\mu \neq \nu$)

$$\sum_{j=0}^{N} e^{i(\mu-\nu)x_j} \Delta x = \sum_{j=0}^{N} \left( e^{i(\mu-\nu)\Delta x} \right)^j \Delta x = \frac{\left( e^{i(\mu-\nu)\Delta x} \right)^{N+1} - 1}{e^{i(\mu-\nu)\Delta x} - 1} = \frac{e^{i(\mu-\nu)2\pi} - 1}{e^{i(\mu-\nu)\Delta x} - 1} = 0$$

and (in the case of $\mu = \nu$)

$$\sum_{j=0}^{N} \Delta x = (N + 1)\Delta x = 2\pi.$$
This property shows that the vectors $v_\omega \in \mathbb{C}^{N+1}$ form an orthogonal basis for $\mathbb{C}^{N+1}$ and thus the expression $f_i = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} \hat{f}_\omega e^{i\omega x_j}$ is always possible for any $f \in \mathbb{C}^{N+1}$. Furthermore, we can determine the $\hat{f}_\mu$ from

$$
\sum_j e^{i\mu x_j} f_j \Delta x = (e^{i\mu x}, \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} \hat{f}_\omega e^{i\omega x}) \Delta x = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} (e^{i\mu x}, \hat{f}_\omega e^{i\omega x}) \Delta x = \sqrt{2\pi} \hat{f}_\mu
$$

or

$$
\hat{f}_\mu = \frac{\sum_j e^{i\mu x_j} f_j \Delta x}{\sqrt{2\pi}}.
$$

3 Parseval’s relation

There is a convenient relation between the discrete two norm and the two norm of the trigonometric basis

$$
|f|_{2,\Delta x} = \sqrt{\langle f, f \rangle_{\Delta x}} = \sqrt{\sum_{\omega=-N/2}^{N/2} |\hat{f}_\omega|^2}.
$$

We can see this directly from

$$
\langle f, f \rangle_{\Delta x} = \frac{\Delta x}{2\pi} \sum_{j=0}^{N} \left( \sum_{\omega=-N/2}^{N/2} \hat{f}_\omega e^{-i\omega x_j} \right) \left( \sum_{\mu=-N/2}^{N/2} \hat{f}_\mu e^{i\mu x_j} \right) = \frac{\Delta x}{2\pi} \sum_{\omega=-N/2}^{N/2} \sum_{\mu=-N/2}^{N/2} \sum_{j=0}^{N} \hat{f}_\omega \hat{f}_\mu e^{i(\mu-\omega)x_j}
$$

$$
= \sum_{\omega=-N/2}^{N/2} \hat{f}_\omega \hat{f}_\omega
$$