1 Pen and paper

1. Consider a two-dimensional periodic grid function. That is, given $u_{ij}$ for $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, N$, or

$$
\mathbf{u} = \begin{pmatrix} u_{00} \\ u_{10} \\ \vdots \\ u_{NN} \end{pmatrix} \in \mathbb{R}^{(N+1)^2}
$$

we can define the periodic $u_{ij} = u_{\text{mod}(i,N+1)\text{mod}(j,N+1)}$ for all $i, j \in \mathbb{Z}$. This is the two-dimensional version of the discussion in the handout on the course page. Assume $N$ is even for simplicity. Define an orthogonal basis $v_{\omega\mu} \in \mathbb{C}^{(N+1)^2}$ for $\omega \in \{-N/2, -N/2 + 1, \ldots, N/2\}$ and $\mu \in \{-N/2, -N/2 + 1, \ldots, N/2\}$ where

$$
\mathbf{u} = \frac{1}{2\pi} \sum_{\omega = -N/2}^{N/2} \sum_{\mu = -N/2}^{N/2} \hat{u}_{\omega\mu} v_{\omega\mu}
$$

and we retain Parseval’s relation

$$
|\mathbf{u}|_{2,\Delta x} = \sqrt{\sum_{\omega = -N/2}^{N/2} \sum_{\mu = -N/2}^{N/2} |\hat{u}_{\omega\mu}|^2}
$$

with

$$
|\mathbf{u}|_{2,\Delta x} = \sqrt{(\mathbf{u}, \mathbf{u})_{2,\Delta x}}, \quad (\mathbf{u}, \mathbf{v})_{2,\Delta x} = \sum_{i=0}^{N} \sum_{j=0}^{N} u_{ij} v_{ij} \Delta x^2
$$

2. Consider the wave equation with periodic boundary conditions in two dimensions

$$
u_t + a_x \cdot \nabla u = u_t + a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial y} = 0, \quad x \in [0, 2\pi)_p, \ y \in [0, 2\pi)_p, \ t \in (0, T)
$$

$$
u(x, y, 0) = u^0(x, y), \quad x \in [0, 2\pi)_p, \ y \in [0, 2\pi)_p.
$$

Solve the problem in terms of the initial data.

3. Consider the upwinding scheme for this problem

$$
\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} + a_x \frac{u_{ij}^n - u_{ij-1}^n}{\Delta x} + a_y \frac{u_{ij}^n - u_{ij-1}^n}{\Delta y} = 0
$$

$$
u_{ij}^0 = u^0(x_i, y_j)
$$

where we assume $a_x > 0$, $a_y > 0$, $\Delta x = \Delta y = \frac{2\pi}{N+1}$ and $\Delta t = \frac{T}{M-1}$. Also, $x_i = i\Delta x$, $y_j = j\Delta y$ for $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, N$ and $n = 0, 1, \ldots, M - 1$. 

1
(a) CFL: Prove that with \( \lambda = \frac{\Delta t}{\Delta x} \), \( 1 - (a_x + a_y)\lambda \in [0, 1) \), \( a_x \lambda \in [0, 1] \) and \( a_y \lambda \in [0, 1] \) are necessary for convergence.

(b) Given \( a_x, a_y \) and \( N \), show how to choose \( M \) so that the condition in part (a) is satisfied.

(c) Prove that this method converges in the infinity norm if the CFL condition is satisfied.

(d) Prove that this method converges in the discrete 2-norm defined in Problem 1 if the CFL condition is satisfied.

(e) Derive a modified equation that this scheme converges to with higher order accuracy (than to the original wave equation).

2 Programming

1. Consider the wave equation with periodic boundary conditions in two dimensions

\[
\frac{\partial u}{\partial t} + a \cdot \nabla u = u_t + a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial y} = 0, \quad x \in [0, 2\pi)_p, \ y \in [0, 2\pi)_p, \ t \in (0, T)
\]

\[
u(x, y, 0) = u^0(x, y), \quad x \in [0, 2\pi)_p, \ y \in [0, 2\pi)_p.
\]

Implement the upwinding scheme from the Pen and Paper problem with \( T = 1 \), \( a_x = \frac{1}{2} \), \( a_y = \frac{\sqrt{3}}{2} \), \( N = 150 \), \( \Delta x = \Delta y = \frac{2\pi}{N+1} \) and

\[
u^0(x, y) = \begin{cases}
1, & x \in (\pi - .125(2\pi), \pi + .125(2\pi)) \text{ and } y \in (\pi - .125(2\pi), \pi + .125(2\pi)) \\
0, & \text{otherwise}
\end{cases}
\]

(a) Choose \( M \) such that the CFL condition is satisfied. Plot the upwinding approximation at times 0, .5 and 1.

(b) Choose \( M \) small enough that the CFL condition is not satisfied. Plot the upwinding approximation at times 0, .5 and 1.

(c) Generalize the upwinding scheme to a semi-Largangian scheme (analogous to 1D) and repeat parts (a) and (b) (i.e. run with those same choices for \( M \) and plot the solution at the same times).