Homework 5  
Math 269B: Numerical methods for PDEs  
Due: Thursday, February 12

1 Pen and paper

1. Consider the wave equation with periodic boundary conditions

\[ u_t + au_x = 0, \quad x \in [0,1)_p, \quad t \in (0,T) \]

\[ u(x,0) = u^0(x), \quad x \in [0,1)_p. \]

(a) Consider the Lax-Wendroff scheme with \( \Delta t = C \Delta x \) where \( C < \frac{1}{|a|} \). Determine \( \hat{C} \) such that for \( \Delta x \) sufficiently small, \( |e^n|_{\Delta x,2} \leq \hat{C} \Delta x^2 \) for \( 0 \leq n \leq M - 1 \) where \( \Delta t = \frac{T}{M-1} \), \( e^n_i = u(x_i,t^n) - u^n_i \) is the error and \( |e^n|_{\Delta x,2} = \sqrt{\sum_i (e^n_i)^2 \Delta x} \). You can assume the solution \( u \) is smooth with bounded derivatives.

(b) Repeat part (a) but with the Crank-Nicholson scheme.

2. Strikwerda problem 3.2.3.

2 Programming

1. Consider the wave equation with periodic boundary conditions

\[ u_t + au_x = 0, \quad x \in [0,1)_p, \quad t \in (0,T) \]

\[ u(x,0) = u^0(x), \quad x \in [0,1)_p. \]

Implement the Lax-Wendroff and Crank-Nicholson schemes with \( T = 1, \quad a = 1, \quad N = 150, \quad \Delta x = \frac{1}{N-1}, \quad \Delta t = \frac{T}{M-1} \) and \( u^0(x) = \begin{cases} 1, & x \in (.375,.625) \\ 0, & \text{otherwise} \end{cases} \).

(a) Use \( M = 200 \).

(b) Use \( M = 100 \).

(c) Plot the Lax-Wendroff approximation next to the Lax-Friedrichs and upwinding approximations with \( N = 15 \) and \( M = 20 \).

(d) Plot the Crank-Nicholson solution next to the backward time/central space approximation with \( N = 15 \) and \( M = 10 \).