Homework 4
Math 269B: Numerical methods for PDEs
Due: Thursday, February 5

1 Pen and paper

1. Consider the wave equation with periodic boundary conditions

\[ u_t + au_x = 0, \ x \in [0,1)_p, \ t \in (0,T) \]

\[ u(x,0) = u^0(x), \ x \in [0,1)_p. \]

(a) Prove that semi-Lagrangian converges (in the infinity norm) even if \( \exists C > 0 \) such that \( C \Delta x < \Delta t \) and even if \( \Delta t > \frac{\Delta x}{a} \).

(b) Generalize the upwinding scheme to

\[ u^{n+1}_i = w^n_{i-1} u^n_{i-1} + w^n_i u^n_i + w^n_{i+1} u^n_{i+1} \]

where the weights \( w^n_{i-1}, w^n_i, w^n_{i+1} \) are from quadratic B-spline interpolation at the point \( x_i - a\Delta t \) and \( x_i^* \) is such that \( x_i - a\Delta t \in (x_i^* - \frac{\Delta x}{2}, x_i^* + \frac{\Delta x}{2}) \)

(c) Prove that the scheme in part (b) converges (in the infinity norm) for this scheme as long as \( \exists C > 0 \) such that \( C \Delta x < \Delta t \).

2. Strikwerda Problem 2.2.1.

3. Strikwerda Problem 2.2.4.

4. Strikwerda Problem 2.2.5.

2 Programming

1. Consider the wave equation with periodic boundary conditions

\[ u_t + au_x = 0, \ x \in [0,1)_p, \ t \in (0,T) \]

\[ u(x,0) = u^0(x), \ x \in [0,1)_p. \]

Implement the schemes from the Pen and Paper problem with \( T = 1, \ a = 1, \ N = 150, \)
\( \Delta x = \frac{1}{N+1}, \ \Delta t = \frac{T}{M+1} \) and \( u^0(x) = \begin{cases} 1, & x \in (.375,.625) \\ 0, & \text{otherwise} \end{cases} \).

(a) Use \( M = 200 \).

(b) Use \( M = 100 \).

(c) Use \( M = 2 \), explain what you see.