1 Pen and paper

1. Consider the wave equation with periodic boundary conditions

\[ u_t + au_x = 0, \quad x \in [0, 1)_p, \quad t \in (0, T) \]

\[ u(x, 0) = u^0(x), \quad x \in [0, 1)_p. \]

(a) Prove that Lax-Friedrichs converges (in the infinity norm) for this scheme as long as \( \exists C > 0 \) such that \( C \Delta x < \Delta t < \frac{\Delta x}{|a|} \).

(b) Derive a scheme

\[ u_{i}^{n+1} = w_{i-1}^{n} u_{i-1}^{n} + w_{i}^{n} u_{i}^{n} + w_{i+1}^{n} u_{i+1}^{n} \]

where the weights \( w_{i-1}^{n}, w_{i}^{n}, w_{i+1}^{n} \) are from quadratic B-spline interpolation at the point \( x_i - a \Delta t \) and \( \Delta t < \frac{\Delta x}{2|a|} \).

(c) Prove that the scheme in part (b) converges (in the infinity norm) for this scheme as long as \( \exists C > 0 \) such that \( C \Delta x < \Delta t < \frac{\Delta x}{2|a|} \).

2 Programming

1. Consider the wave equation with periodic boundary conditions

\[ u_t + au_x = 0, \quad x \in [0, 1)_p, \quad t \in (0, T) \]

\[ u(x, 0) = u^0(x), \quad x \in [0, 1)_p. \]

Implement the schemes from the Pen and Paper problem with \( T = 1, \quad a = 1, \quad N = 150, \quad \Delta x = \frac{1}{N-1}, \quad \Delta t = \frac{T}{M-1} \) and \( u^0(x) = \begin{cases} 1, & x \in (.375, .625) \\ 0, & \text{otherwise} \end{cases} \).

(a) Use \( M = 200 \).

(b) Use \( M = 100 \).

2. Repeat Problem 1 with the following scheme

\[ \frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} + a \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = 0 \]