Take Home Final: Math 269B

- 1. Show that the Crank-Nicholson scheme for the scalar heat equation $u_t = bu_{xx}$ satisfies

$$\|v^{n+1}\|_\infty \leq \|v^n\|_\infty$$

for all solutions if and only if $b\mu \leq 1$.

- 2. Show that the DuFort-Frankel Scheme for $u_t = b\left(u_{xx} + u_{yy}\right)$ is unconditionally stable when $\Delta x = \Delta y$.

- 3. Consider the second order wave equation $Mu_{tt} = au_{xx} + bu_{tx}$. This equation describes a damped elastic material and can be equivalently written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v \\ \frac{a}{M}u_{xx} + \frac{b}{M}v_x \end{pmatrix}.$$

a. Derive a time step restriction that will guarantee stability for the following scheme

$$v_m^{n+\frac{1}{2}} = v_m^n + \frac{\Delta t}{2}\left(\frac{a}{M}\left(\frac{u_{m+1}^n + u_{m-1}^n - 2u_m^n}{\Delta x^2}\right) + \frac{b}{M}\left(\frac{v_{m+1}^n - v_{m-1}^n}{2\Delta x}\right)\right)$$

$$u_m^{n+1} = u_m^n + \Delta t v_m^{n+\frac{1}{2}}$$

$$v_m^{n+1} = v_m^{n+\frac{1}{2}} + \frac{\Delta t}{2}\left(\frac{a}{M}\left(\frac{u_{m+1}^{n+1} + u_{m-1}^{n+1} - 2u_m^{n+1}}{\Delta x^2}\right) + \frac{b}{M}\left(\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2\Delta x}\right)\right).$$

b. Derive a time step restriction that will guarantee stability for the following scheme (backward time/central space)

$$u_m^{n+1} = u_m^n + \Delta t v_m^{n+1}$$

$$v_m^{n+1} = v_m^n + \Delta t\left(\frac{a}{M}\left(\frac{u_{m+1}^{n+1} + u_{m-1}^{n+1} - 2u_m^{n+1}}{\Delta x^2}\right) + \frac{b}{M}\left(\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2\Delta x}\right)\right).$$

c. Provide a bound for

$$\|\mathbf{e}^n\|_{\Delta x,2} = \sqrt{\sum_{m=-\infty}^{\infty} |\mathbf{e}_m^n|^2 \Delta x}$$

- 4. How large can ϵ be for upwinding with negative viscosity? I.e. which ϵ still yield a stable scheme for fixed $\lambda = \frac{\Delta t}{\Delta x}$?

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + a\frac{v_m^n - v_{m-1}^n}{\Delta x} + \Delta t\epsilon\frac{v_{m+1}^n + v_{m-1}^n - 2v_m^n}{\Delta x^2} = 0$$

- 5. Write a program showing the effect of ϵ for problem 4. Use piecewise constant initial data.

- 6. Show that the nine point Laplacian satisfies a discrete maximum principle.

- 7. Write a program that approximates the two dimensional advection/diffusion equation on a periodic domain using forward time and central space.

$$u_t + a(x,y)u_x + b(x,y)u_y = c(u_{xx} + u_{yy}).$$

  Use $a = sin(x)cos(y)$, $b = -cos(x)sin(y)$, $c = 10$. Also, use the periodic spatial domain $[0, 2\pi]$x$[0, 2\pi]$ and use $u(x,y) = sin(x)cos(y)$ as your initial conditions.

- 8. Extra credit: Use your code from problem 7 and a periodic Poisson solver to simulate incompressible flow (see handout for more details). Use the same periodic domain as in problem 7 and the initial conditions $u(x,y) = sin(x)cos(y)$ and $v(x,y) = -cos(x)sin(y)$.