M269B: HOMEWORK 3

Due Monday, Feb 2

Strickwerda: 3.1.1, 3.2.3, 4.1.2

Computational

Consider the following initial boundary value problem for the one-way wave equation:

$$u_t + au_x = xe^{-t}, \quad x \in (0, 2], t \in (0, 1] \tag{1}$$

$$u(x, 0) = (1 - x) \tag{2}$$

$$u(0, t) = (1 - t). \tag{3}$$

Numerically solve the above using Lax-Wendroff and Crank-Nicolson schemes for $\Delta x = 1/25$ and $a = 1, 10, 100$. For the Lax-Wendroff scheme, use $a\lambda = .8$. For the Crank-Nicolson scheme use $\Delta t = 1/25$. Plot the error at $t = 1$ for both schemes for all values of a and for each value of a plot the exact solution. Also include the time required to compute the solution to each problem with each scheme. With this information, which scheme do you think was efficient and for what values of a?

In MATLAB you can time things like this:
tic;
...computation...
finaltime = toc;

Derivation of exact solution to (1):
The characteristics $(x(\tau), t(\tau))$, where τ parametrizes the characteristic curves, satisfy the following equations

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = a, \quad \frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau} = 1, \tag{4}$$

$$x_0 = x(0), \quad t_0 = t(0). \tag{5}$$

Along the characteristics $u(x, t)$ satisfies

$$\frac{\mathrm{d}u(x(\tau), t(\tau))}{\mathrm{d}\tau} = f(x(\tau), t(\tau)) = xe^{-t}. \tag{6}$$

First condisider $x \geq at$. In this region the characteristics originate from the x-axis so $t_0 = 0$, thus $t(\tau) = \tau$ and $x(\tau) = x(t) = at + x_0$. Given x, t we can solve for $x_0 = x - at$. From this and (6) we have that

$$u(x(\tau), t(\tau)) = u(x_0, t_0) + \int_0^t f(as + x - at, s)ds = 1 + at - x + \int_0^t (x - at + as)e^{-s}ds. \tag{7}$$

Using that

$$\int_0^t se^{-s} = (1 - e^{-t}(t + 1)), \tag{8}$$

we get

$$u(x, t) = 1 - (x - at)(e^{-t}) + a(1 - e^{-t}(1 + t)). \tag{9}$$

With some effort it can be verified that this solves the IVP for $x \geq at$.

For the domain $x \leq at$ we have that characteristics originate from the t-axis, so $x_0 = 0$, $x(\tau) = a\tau$ and $t(\tau) = \tau + t_0 = x/a + t_0$. Using this, the ODE (6) and that $t_0 = t - x/a$ we have

$$u(x(\tau), t(\tau)) = u(x_0, t_0) + \int_0^{x/a} f(as, s + t - x/a)ds = 1 + x/a - t + \int_0^{x/a} ase^{-s+(x/a-t)}ds. \quad (10)$$

Integrating (correctly!) should give

$$u(x, t) = 1 + x/a - t + ae^{x/a-t}(1 - e^{-x/a}(\frac{x}{a} + 1)). \quad (11)$$