Homework 3
Math 269A: Numerical methods for ODEs
Due: Wed, November 29

1 Pen and paper

1. Solve the difference equation

\[ z^{n+1} - 4z^n + 4z^{n-1} = 0, \quad n = 1, 2, \ldots \]
\[ z^0 = 1, \quad z^1 = 4 \]

2. Solve the difference equation

\[ z^{n+1} - 9z^n + 27z^{n-1} - 27z^{n-2} = 0, \quad n = 2, 3, \ldots \]
\[ z^0 = 1, \quad z^1 = 6, \quad z^2 = 45 \]


2 Programming

1. Consider the problem

\[ \frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{m} \frac{\partial e}{\partial x}(x(t)) \\ \end{pmatrix}, \quad t > 0, \quad \text{and} \quad x(0) = 2, \quad v(0) = 0 \]

where \( m > 0 \). Write a program that approximates the solution with \( T = 2\pi \), \( N_f = 3, 30, 300, 3000 \). For problems (a) and (b), estimate the error of your method by plotting

\[ \log(|E|_{\infty}) \text{ vs } \log(\Delta t) \]

where \( E = \begin{pmatrix} e^0 \\ \vdots \\ e^{N_f} \end{pmatrix} \) and \( e^n \) is the error in your approximation at time \( t^n = n\Delta t \). Do this using:

(a) \( e(x) = \frac{k}{2} (x - 1)^2 \) and Adams-Bashforth, \( k = 2 \).
(b) \( e(x) = \frac{k}{2} (x - 1)^2 \) and Adams-Moulton, \( k = 2 \).
(c) \( e(x) = \frac{k}{2} \log(x)^2 \) and Adams-Bashforth, \( k = 3 \).
(c) \( e(x) = \frac{k}{2} \log(x)^2 \) and BDF, \( k = 2 \).

2. Visualize the absolute stability regions (assuming \( \text{Re}(\lambda) < 0 \)) for Adams-Bashforth, \( k = 1 - 4 \), Adams-Moulton, \( k = 1 - 4 \) and BDF, \( k = 1 - 4 \) by looking for all roots \( \xi \) of \( \frac{\rho(\xi)}{\sigma(\xi)} = \Delta t \lambda \) with \( |\xi| \geq 1 \). Here

\[ N^n(Z) = \sum_{j=0}^{k} \alpha_j z^{n+1-j} - \Delta t \sum_{j=0}^{k} \beta_j \lambda z^{n+1-j} \]  
(1)

\[ \rho(\xi) = \sum_j \alpha_{k-j} \xi^j, \quad \sigma(\xi) = \sum_j \beta_{k-j} \xi^j \]  
(2)