1 Pen and paper

1. Consider the problem
\[
\frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \left( -\frac{1}{m} \frac{\partial e}{\partial x}(x(t)) \right), \quad t > 0, \quad \text{and} \quad x(0) = x_0, \ v(0) = v_0
\]
where \( m > 0 \). Linearize the problem around points \( \left( \hat{x}, \hat{v} \right) \in \mathbb{R}^2 \) and label the linearized problem as either stable, asymptotically stable, or unstable for
(a) \( e(x) = \frac{k}{2} (x - 1)^2 \).
(b) \( e(x) = \frac{k}{2} \log(x)^2 \).
(c) \( e(x) = \frac{k}{8} (x^2 - 1)^2 \). In each of these you should assume \( k > 0 \).

2. Solve the ODE
\[
\frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \left( -\frac{k}{m} (x(t) - 1) - \gamma v(t) \right), \quad t > 0, \quad \text{and} \quad x(0) = x_0, \ v(0) = v_0
\]
where \( k, m, \gamma > 0 \).

3. Define a Lyapunov function for Problem 2.

4. Consider the mass/spring problem
\[
\frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \left( -M^{-1} \frac{\partial e}{\partial x}(x(t)) \right), \quad t \in (0, T)
\]
where
\[
\begin{pmatrix} x(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} x^0 \\ v^0 \end{pmatrix}
\]
and
\[
x(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix} \in \mathbb{R}^4, \quad x_i(t) = \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} \in \mathbb{R}^2
\]
\[
v(t) = \begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix} \in \mathbb{R}^4, \quad v_i(t) = \begin{pmatrix} u_i(t) \\ v_i(t) \end{pmatrix} \in \mathbb{R}^2
\]
\[
M = \begin{pmatrix} m_0 I_2 & 0_2 \\ 0_2 & m_1 I_2 \end{pmatrix} \in \mathbb{R}^{4 \times 4}
\]
where \( I_2 \) is the 2 \times 2 identity matrix in \( 0_2 \) is the 2 \times 2 matrix of zeros.

(a) Let \( e(x) = \frac{k}{2} l_0 \left( \frac{1}{l_0} - 1 \right)^2 \), \( l(x) = |x_1 - x_0| \), \( l_0 = |x_1(0) - x_0(0)| \). Does this problem satisfy the hypothesis of Theorem 1.1?

(b) Let \( e(x) = \frac{k}{2} \gamma^2 \). Does this problem satisfy the hypothesis of Theorem 1.1?

(c) Let \( e(x) = \frac{k}{2} \gamma^2 \), \( x_1 = v_1 = 0 \) and \( m_1 = \infty \) (i.e. \( M^{-1} = \begin{pmatrix} \frac{1}{m_0} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \)). Solve the ODE for this case.
5. Problem 2.3 in Ascher and Petzold.
6. Problem 3.1 from Ascher and Petzold.
7. Consider the ODE
\[
\frac{\partial Y(t)}{\partial t} = A(t)Y(t), \ t > 0 \\
Y(0) = I
\]
Show that if \( \exists M > 0 \) such that \( |A(t)|_F \leq M \) for all \( t \) and \( A \) is continuous, then \( \det(Y(t)) > 0 \).

2 Programming

1. Consider the particle kinematics with
\[
\frac{\partial}{\partial t} x = \begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix} x, \ t \in (0, 2\pi) \\
x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
Write a program to approximate the solution of this equation with Forward Euler. Compare your Forward Euler approximation to the exact solution for \( \epsilon = .1, -.1 \) and \( N_f = 5, 10, 20 \).

2. Write a program to approximate the mass/spring problem in Problem 2 with forward Euler. Use \( N_f = 100, T = 8, \Delta t = \frac{T}{N_f}, k = 2, m = 1, \gamma = 0, x_0 = 2 \) and \( v_0 = 0 \).

3. Write a program to approximate the mass/spring problem in Problem 2 with backward Euler. Use \( N_f = 100, T = 8, \Delta t = \frac{T}{N_f}, k = 2, m = 1, \gamma = 0, x_0 = 2 \) and \( v_0 = 0 \).

4. Derive error bounds for the methods in Programming Problems 1-3 and verify that your results satisfy the bounds.