1 Pen and paper

1. Runge-Kutta: Consider the RK2 scheme

\[ N_n^N(Z) = \frac{z^{n+1} - z^n}{\Delta t} - \frac{1}{2} f(z^n, t^n) - \frac{1}{2} f(z^n + \Delta t f(z^n, t^n), t^{n+1}) \]

applied to the test equation

\[ \frac{\partial y}{\partial t}(t) = f(y(t), t) = \lambda y(t), \quad t > 0 \]

\[ y(0) = y_0. \]

If \( \text{Re}(\lambda) < 0 \), how big should \( \Delta t \) be to guarantee that \( |z^{n+1}| \leq |z^n| \)?

2. Runge-Kutta: RK3

(a) Let \( \hat{y}^{n+1} = y^n + \Delta t f(y^n, t^n) \). Show that

\[ \frac{f(\hat{y}^{n+1}, t^{n+1}) - f(y^n, t^n)}{\Delta t} = y''(t^n) + O(\Delta t) \]

where \( y^n = y(t^n) \) and

\[ \frac{\partial y}{\partial t}(t) = f(y(t), t), \quad t > 0 \]

\[ y(0) = y_0. \]

(b) Let \( \hat{y}^{n+\frac{1}{2}} = y^n + \frac{\Delta t}{2} f(y^n, t^n) \). Show that

\[ y^{n+\frac{1}{2}} = \hat{y}^{n+\frac{1}{2}} + \frac{\Delta t^2}{8} \left( \frac{f(\hat{y}^{n+1}, t^{n+1}) - f(y^n, t^n)}{\Delta t} \right) + O(\Delta t^3) \]

and

\[ y^{n+1} = \hat{y}^{n+1} + \frac{\Delta t^2}{2} \left( \frac{f(\hat{y}^{n+1}, t^{n+1}) - f(y^n, t^n)}{\Delta t} \right) + O(\Delta t^3) \]

(c) Combine these ideas with Simpson’s rule

\[ y^{n+1} - y^n = \frac{\Delta t}{6} \left( f(y^n, t^n) + 4 f(y^{n+\frac{1}{2}}, t^{n+\frac{1}{2}}) + f(y^{n+1}, t^{n+1}) \right) + O(\Delta t^5) \]

to define a third order Runge-Kutta method.

(d) Consider the test equation \( (f(y(t), t) = \lambda y(t)) \). Let \( \xi = \lambda \Delta t \). For which \( \xi \) is your scheme A-stable?

3. Ascher and Petzold, Problems 4.2 and 4.3.
4. Multi-step: Consider the scheme
\[ N^N_n(Z) = \frac{z^{n+1} - z^{n-1}}{2\Delta t} - f(z^n, t^n) \]
applied to the test equation
\[ \frac{\partial y}{\partial t}(t) = f(y(t), t) = \lambda y(t), \quad t > 0 \]
\[ y(0) = y_0. \]
Let \( \xi = \lambda \Delta t \). For which \( \xi \) is your scheme A-stable?

2 Programming

1. Let \( \xi_{ij} = ax_i + \hat{b}y_j \) for \( i = 0, 1, \ldots, M - 1 \) and \( j = 0, 1, \ldots, N - 1 \). Visualize the stability regions of RK2 and RK3 from the pen and paper problems by creating an image with pixel \( i, j \) color coded based on A-stability of the respective schemes for the given value of \( \xi_{ij} \). You should choose appropriate \( a, b, M, N \).

2. Mass/spring: Consider the system
\[ \frac{\partial}{\partial t} \left( \begin{array}{c} x(t) \\ v(t) \end{array} \right) = \left( \begin{array}{c} \frac{1}{m} \left( -\frac{\partial^2}{\partial x^2}(x(t)) - \gamma v(t) \right) \end{array} \right), \quad t > 0 \]
with \( e(x) = \frac{k}{2} x^2 \), \( m > 0 \), \( k > 0 \), \( \gamma > 0 \) and initial conditions \( x(0) = 1, v(0) = 0 \).

   (a) Let \( m = 1, k = 100, T = 10 \) and \( \gamma = \frac{3}{2} \sqrt{4km} \). Plot the Forward Euler approximation of the problem with \( \Delta t = \frac{3}{2} \min(\frac{-2a_1}{a_1^2 + b_1^2}, \frac{-2a_2}{a_2^2 + b_2^2}) \) \( \lambda_i = a_i + \hat{b}_i \) are the eigenvalues of the mass/spring matrix.

   (b) Let \( m = 1, k = 100, T = 10 \) and \( \gamma = \frac{3}{2} \sqrt{4km} \). Plot the Forward Euler approximation of the problem with \( \Delta t = \frac{2}{3} \min(\frac{-2a_1}{a_1^2 + b_1^2}, \frac{-2a_2}{a_2^2 + b_2^2}) \) \( \lambda_i = a_i + \hat{b}_i \) are the eigenvalues of the mass/spring matrix.

   (c) Repeat (a) and (b) with Backward Euler. Comment on your results.

3. Repeat Problem 2 with \( e(x) = \frac{k}{2} (\ln(x))^2 \), \( x(0) = 2, v(0) = 0 \). However, rather than using \( \Delta t = \min(\frac{-2a_1}{a_1^2 + b_1^2}, \frac{-2a_2}{a_2^2 + b_2^2}) \) as the critical time step size for stability, you should suggest one.