Difference Equation Notes

1 Multistep for test equation

Multistep methods for the test equation have the following form

\[ \mathcal{L}_n^N(Z) = \sum_{j=0}^{k} \alpha_j z^{n+1-j} - \Delta t \sum_{j=0}^{k} \beta_j \lambda z^{n+1-j} = \sum_{j=0}^{k} a_j z^{n+1-j}, \quad n = k - 1, k, \ldots, N - 1 \]

with \( a_j = \alpha_j - \Delta t \lambda \beta_j \) and \( Z \in \mathbb{C}^{N+1} \). Furthermore, we can say \( a_0 \neq 0 \) or else the multistep scheme would be useless in marching our discrete solution forward in time. Therefore,

\[ z^{n+1} = -\frac{1}{a_0} \sum_{j=1}^{k} a_j z^{n+1-j} = \sum_{j=1}^{k} \hat{a}_j z^{n+1-j} \]

with \( \hat{a}_j = -\frac{a_j}{a_0} \).

2 Solutions of difference equation

Denote a solution \( Z_i \) to the equations \( \mathcal{L}_n^N(Z_i) = 0 \), \( n = k - 1, k, \ldots, N - 1 \) as

\[ Z_i = \begin{pmatrix} z_0^i \\ z_1^i \\ \vdots \\ z_N^i \end{pmatrix} \]

and use \( \tilde{z}_i^n \) to denote

\[ \tilde{z}_i^n = \begin{pmatrix} z_{i-k}^n \\ z_{i-k+1}^n \\ \vdots \\ z_i^n \end{pmatrix}. \]

With this notation, we have

\[ a^T \tilde{z}_i^n = 0, \quad n = k, k+1, \ldots, N \]

where

\[ a = \begin{pmatrix} a_k \\ a_{k-1} \\ \vdots \\ a_0 \end{pmatrix}. \]

2.1 Linear dependence of \( \tilde{z}_i^n \) → linear dependence of \( Z_i \)

Suppose we have solutions \( Z_i \) for \( i = 1, 2, \ldots, l \) with \( c_i \) such that

\[ \tilde{z}_i^k = \sum_{i=1}^{l-1} c_i \tilde{z}_i^k. \]
Then,
\[
z_{l}^{k+1} = \sum_{j=1}^{k} \hat{a}_j z_{l}^{k+1-j} = \sum_{j=1}^{k} \hat{a}_j \sum_{i=1}^{l-1} c_i z_i^{k+1-j} = \sum_{i=1}^{l-1} c_i \sum_{j=1}^{k} \hat{a}_j z_i^{k+1-j} = \sum_{i=1}^{l-1} c_i z_i^{k+1}
\]
and thus since we can keep repeating this argument, we see that it must be true that

\[
Z_l = \sum_{i=1}^{l-1} c_i Z_i
\]

2.2 If \( Z_i \) is a (non-zero) solution, then \( \tilde{z}_i^k \neq 0 \)
If \( \tilde{z}_i^k = 0 \), then \( z_i^{k+1} = \sum_{j=1}^{k} \hat{a}_j z_i^{k+1-j} = 0 \). And similarly, \( z_i^{k+2} = 0, \ldots \) so if \( \tilde{z}_i^k = 0 \) then \( Z_i = 0 \) and we are assuming that isn’t true.

2.3 If \( Z_i, i = 1, 2, \ldots, M \) are linearly independent (non-zero) solutions, then \( \tilde{z}_i^n, i = 1, 2, \ldots, M \) are linearly independent and non-zero.
This is implied by the results of Sections 2.1 and 2.2.

2.4 There are at most \( k \) linearly independent \( \tilde{z}_i^n \) with \( a^T \tilde{z}_i^n = 0 \)
Since \( a \in \mathbb{C}^{k+1} \), the set of \( x \in \mathbb{C}^{k+1} \) with \( a^T x = 0 \) has dimension \( k \). If there were more than \( k \) linearly independent \( \tilde{z}_i^n = 0 \), this would contradict that.

2.5 There are at most \( k \) linearly independent \( Z_i \)
This holds from the results of Sections 2.4 and 2.3