Let $S$ be a nonempty linearly independent set of a vector space $V$. Prove that for any nonzero vector $v \in V$, there is an $x \in S$ such that $S \setminus \{x\} \cup \{v\}$ is linearly independent.

**Solution.** Let $v \in V$ be nonzero.

- **Case 1:** Suppose that $v \notin \text{Span}(S)$. Then $S \cup \{v\}$ is linearly independent. Let $x$ be any vector in $S$. Since $S \cup \{v\}$ is linearly independent, $S \setminus \{x\} \cup \{v\}$ is linearly independent, since subsets of linearly independent sets are linearly independent.

- **Case 2:** Suppose that $v \in \text{Span}(S)$. Then there exists $x_1, \ldots, x_n \in S$ such that $v = a_1x_1 + \cdots + a_nx_n$ for some scalars $a_i$. Furthermore, this linear combination is unique, by linear independence of $S$. Since $v$ is nonzero, at least one of the $a_i$'s is nonzero. Since addition in $V$ is commutative, we may assume by reordering that $a_1 \neq 0$.

We claim that $S \setminus \{x_1\} \cup \{v\}$ is linearly independent. To show this, we must demonstrate that every finite subset of $S \setminus \{x_1\} \cup \{v\}$ is linearly independent.

Because subsets of $S \setminus \{x_1\}$ are linearly independent by assumption ($S \setminus \{x_1\}$ is a subset of $S$), it suffices to consider a finite subset of $S \setminus \{x_1\} \cup \{v\}$ of the form $\{v, y_1, \ldots, y_k\}$ where $\{y_1, \ldots, y_k\} \subseteq S \setminus \{x_1\}$.

Suppose that
\[
cv + c_1y_1 + \cdots + c_ky_k = 0.
\]
If $c = 0$, then by linear independence of $\{y_1, \ldots, y_k\}$, $c_1 = \cdots = c_k = 0$. If $c \neq 0$, then
\[
v = -\frac{c_1}{c}y_1 + \cdots + -\frac{c_k}{c}y_k.
\]
This contradicts the fact that $v$ can be uniquely expressed as $a_1x_1 + \cdots + a_nx_n$, since $x_1 \notin \{y_1, \ldots, y_k\}$. Thus, $c = c_1 = \cdots = c_k = 0$, and so $\{v, y_1, \ldots, y_k\}$ is linearly independent.

\qed