115AH EXTRA PROBLEMS

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Here are some supplementary exercises involving linear algebra. I'll state many of them in class, but probably not all of them. They may or may not be challenging and they may or may not be relevant to the immediate 115AH material, but they're all interesting (to me, at least!).

In general, *you should focus on the content covered in class, as well as the assigned homework exercises*. These problems are an extra challenge, and are just here for fun.

April 3, 2018

1. Let
$$\ell^2(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^\infty \ : \ a_n \in \mathbb{R}, \ \sum_{n=1}^\infty |a_n|^2 < \infty \right\}.$$
Endow $\ell^2(\mathbb{N})$ with component-wise addition and scalar multiplication. Show that $\ell^2(\mathbb{N})$ is a vector space over $\mathbb{R}$.

2. Let V be the set of smooth[1] functions $f : \mathbb{R} \to \mathbb{R}$ which solve the differential equation
$$f''(x) \cdot f'(x) = 0.$$
Is V a vector space?

3. Let $C^\infty(\mathbb{R}^3)$ be the set of smooth[2] functions $f : \mathbb{R}^3 \to \mathbb{R}$.
Fix $v \in \mathbb{R}^3$. Let V be the set of functions $D : C^\infty(\mathbb{R}^3) \to \mathbb{R}$ satisfying the following properties:
   (a) For any $f, g \in C^\infty(\mathbb{R}^3)$, $D(f+g) = D(f) + D(g)$.
   (b) For any $f \in C^\infty(\mathbb{R}^3)$ and any $\alpha \in \mathbb{R}$, $D(\alpha f) = \alpha D(f)$.
   (c) For any $f, g \in C^\infty(\mathbb{R}^3)$, $D(fg) = D(f) \cdot g(v) + f(v) \cdot D(g)$.

   Show that V is a vector space over $\mathbb{R}$. Show that D defined by
$$D(f) = \frac{\partial f}{\partial x}(v)$$
is an element of V.

April 5, 2018

1. Recall that
$$\ell^2(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^\infty \ : \ a_n \in \mathbb{R}, \ \sum_{n=1}^\infty |a_n|^2 < \infty \right\}.$$
In class, I also defined
$$\ell^1(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^\infty \ : \ a_n \in \mathbb{R}, \ \sum_{n=1}^\infty |a_n| < \infty \right\}.$$
Both $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$ are vector spaces. Show that $\ell^1(\mathbb{N})$ is a subspace of $\ell^2(\mathbb{N})$.

[1] A function is *smooth* if all orders of derivatives exist and are continuous.
[2] Here, *smooth* means all orders of partial derivatives exist and are continuous.

2. More generally, we can define

$$\ell^p(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^{\infty} \ : \ a_n \in \mathbb{R}, \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

for any positive real number $p > 0$. Show that $\ell^p(\mathbb{N})$ is a vector space. When is $\ell^p(\mathbb{N})$ a subspace of $\ell^q(\mathbb{N})$?

3. Let W_1, W_2, W_3 be subspaces of a *real* vector space V. Show that $W_1 \cup W_2 \cup W_3$ is a subspace of V if and only if one of the subspaces contains the other two.

   Show that this is false if the vector space V is over the field $\mathbb{F}_2 = \{0, 1\}$.

4. Let $\mathbb{F}_3 = \{0, 1, 2\}$ be the field with three elements. Consider the $\mathbb{F}_3$-vector space

$$\mathbb{F}_3^2 = \{ (x_1, x_2) \ : \ x_i \in \mathbb{F}_3 \}.$$

   Write $\mathbb{F}_3^2$ as the union of subspaces, none of which is contained in any other.

April 9, 2018

1. Let $C([0, 2\pi])$ denote the real vector space of continuous functions $f : [0, 2\pi] \to \mathbb{R}$. Show that for all $n \geq 1$, the set of elements

$$\{\cos(x), \cos(2x), \dots, \cos(nx)\}$$

   is linearly independent.

   Conclude that $C([0, 2\pi])$ is an infinite-dimensional vector space.

   [Hint: You could probably brute force this, but I have another solution in mind. What do you know about the integral

$$\int_0^{2\pi} \cos(kx)\cos(jx)\, dx$$

   if $k \neq j$?]

2. Let A be an $n \times n$ real matrix. Suppose that $v_1, v_2, \dots v_k \in \mathbb{R}^n$ are nonzero vectors such that $Av_j = \lambda_j v_j$ for *distinct* scalars $\lambda_1, \dots, \lambda_k$. Show that $\{v_1, \dots, v_k\}$ is a linearly independent set.

April 19, 2018

Let $\mathbb{R}^\infty = \{ (x_0, x_1, x_2, \dots) \ : \ x_i \in \mathbb{R} \}$ be the vector space of real-valued sequences. Define the convolution $a * b$ of two sequences $a = (a_0, a_1, a_2, \dots)$ and $b = (b_0, b_1, b_2, \dots)$ in $\mathbb{R}^\infty$ by[3]

$$(a * b)_k = \sum_{i=0}^{k} a_{k-i} b_i.$$

Then $a * b \in \mathbb{R}^\infty$.

1. Let $\ell^1(\mathbb{N}) = \{ (x_0, x_1, x_2, \dots) \in \mathbb{R}^\infty \ : \ \sum_{n=0}^{\infty} |x_0| < \infty \}$ be the subspace of $\mathbb{R}^\infty$ of absolutely convergent sequences. Show that $\ell^1(\mathbb{N})$ is closed under convolution. That is, if $a, b \in \ell^1(\mathbb{N})$, then $a * b \in \ell^1(\mathbb{N})$.

2. Let $\ell^2(\mathbb{N}) = \left\{ (x_0, x_1, x_2, \dots) \in \mathbb{R}^\infty \ : \ \sum_{n=0}^{\infty} |x_0|^2 < \infty \right\}$ be the subspace of $\mathbb{R}^\infty$ of square-summable sequences. Is $\ell^2(\mathbb{N})$ closed under convolution?

[3] Here, define $a_m = 0$ if $m < 0$.