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**Laumon, Gérard** (F-PARIS11)

★ _Cohomology of Drinfeld modular varieties. Part I._
Geometry, counting of points and local harmonic analysis.

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**Laumon, Gérard** (F-PARIS11)

★ _Cohomology of Drinfeld modular varieties. Part II. (English summary)_
Automorphic forms, trace formulas and Langlands correspondence.
With an appendix by Jean-Loup Waldspurger.

FEATURED REVIEW.
Eichler-Shimura theory describes the action of the Galois group on the $l$-adic cohomology of a modular curve in terms of Hecke correspondences and thus provides a fundamental link between the arithmetic of modular curves and modular forms. A great deal of progress in the theory of automorphic forms over the past thirty years has come from attempts to formulate and prove natural generalizations of Eichler-Shimura theory. Langlands and later Kottwitz developed a beautiful, conjectural formulation of Eichler-Shimura theory for higher-dimensional Shimura varieties using sophisticated notions from representation theory (theory of endoscopy, the Langlands group, Arthur’s extended $L$-packets, etc.). The extension to automorphic forms over function fields was pioneered by Drinfeld in his seminal work on Drinfeld modules and their applications to the
Langlands conjectures for GL(2) over a function field.

The two volumes under review take up the project of extending Drinfeld's theory from GL(2) to GL(n). This is a formidable task which brings together heavy amounts of local and global harmonic analysis on GL(n) with a careful study of the cohomology of the moduli spaces for Drinfeld modules. Fortunately for those wishing to learn this beautiful field, the author has produced an outstanding work which is equally a mathematical and an expository achievement.

Whether one is dealing with Shimura varieties attached to a reductive group G or Drinfeld modular varieties, the basic goal is always the same: to decompose the appropriate l-adic cohomology (e.g., cohomology with compact supports, intersection cohomology, etc.) relative to the commuting actions of the Galois group and the ring of Hecke operators. In fact, after passing to the limit, the Hecke operators give way to a representation of the adelic group \( G(\mathbb{A}^\infty) \). Here \( \mathbb{A}^\infty \) denotes the ring of adeles away from \( \infty \) where \( \infty \) is a set of places of the global field \( F \). If \( F \) is a number field, \( \infty \) is the set of Archimedean places, and in the function field case it is the singleton consisting of the arbitrary place of \( F \) that must be fixed to define Drinfeld modules. The joint action of Galois and Hecke yields a correspondence \( \pi \rightarrow \sigma(\pi) \) between the finite parts \( \pi_f = \bigotimes_{v \neq \infty} \pi_v \) of certain cuspidal representations of \( G(\mathbb{A}) \) and l-adic representations of the Galois group \( \text{Gal}(\overline{F}/F) \). It is then of interest to describe \( \sigma(\pi) \) in terms of \( \pi \).

This goal is achieved in the case at hand, namely for GL(n) over a function field, in Chapter 12 of Volume II. The cuspidal representations \( \pi \) intervening in the description of the cohomology (with compact supports) of the Drinfeld modular varieties are precisely those whose local component at the place \( \infty \) are Steinberg representations. Theorem 12.4.1 asserts that, as expected, the correspondence \( \pi \rightarrow \sigma(\pi) \) is the Langlands correspondence, i.e., it is characterized by the condition that at almost all places \( v \), the characteristic polynomial of the conjugacy class of a Frobenius element coincides, under a suitable normalization, with the characteristic polynomial of the Langlands class attached to the local component \( \pi_v \). The Ramanujan-Petersson conjecture for \( \pi \) is also derived using work of Jacquet-Shalika, Grothendieck and Deligne. We might add that it is precisely this sort of easily stated relation between \( \pi \) and \( \sigma(\pi) \) which does not hold when the group is not GL(n). The correct relation can be formulated only after the introduction of endoscopic groups, and this is what substantially complicates the conjectural statements of Langlands-Kottwitz for general Shimura varieties. The hypothesis that \( \pi_\infty \) is a Steinberg representation is intrinsic to the approach via Drinfeld modules taken by Laumon. To treat arbitrary cuspidal representations, it is necessary to use Drinfeld’s theory of shtukas. This was carried out by Drinfeld himself for GL(2) and is currently being pursued for GL(n) in work of Laurent Lafforgue.

Although Eichler-Shimura theory rests on the congruence relation for Hecke correspondences, most higher-dimensional versions, including the present one, use a different approach originating in work of Ihara and Langlands, namely, a comparison of the Arthur-Selberg trace formula and the Grothendieck-Lefschetz fixed point formula. This method of proof involves two main stages, the first being concerned with preparations for the trace formula and the second with the trace formula itself. The material is correspondingly divided between the two volumes.

The first three chapters of Volume I introduce the Drinfeld modular varieties and derive a
formula, familiar from the case of Shimura varieties, for the Lefschetz numbers of an operator of the form $\text{Frob}_o^r \times T$, where $\text{Frob}_o$ is a power of the Frobenius at an (unramified) place $o$ and $T$ is a Hecke operator away from $o$ and $\infty$. Laumon first writes the formula as a sum over an appropriate set of elliptic elements of certain terms, each of which is a product of volume factors, a twisted orbital integral at $o$, and an orbital away from $o$ and $\infty$. The final formula only involves ordinary orbital integrals, but the idea of introducing the twisted orbital integral at this point is due to Kottwitz (in the Shimura variety case), who realized that a suitable case of the Fundamental Lemma (as formulated by Langlands) would then show that the twisted orbital integral is equal to an ordinary orbital integral. The case of the Fundamental Lemma needed here is proved in Chapter 4. The rest of Volume I is devoted to the construction of a particular function $f_{\infty}$ on $GL_n(F_{\infty})$ whose orbital integrals give the $\infty$-adic contribution to the volume factors. Laumon introduces the notion of a very cuspidal function and proves that the $f_{\infty}$ he has constructed is very cuspidal and is also a pseudo-coefficient for the Steinberg representation. The end result is that the Lefschetz numbers are expressed as global orbital integrals of a certain function $f_A$ on $GL_n(A)$. Volume I also contains an excellent discussion of the unramified principal series and four useful appendices devoted to simple algebras, Dieudonné modules, some combinatorics, and representation theory.

Most of Volume II is devoted to computing what happens when you plug $f_A$ into Arthur’s non-invariant trace formula. This trace formula asserts the equality of two complicated expressions called the geometric and spectral sides of the trace formula. Although the non-invariant trace formula depends on the choice of an arbitrary truncation parameter, Laumon develops the trick of Kazhdan, which asserts that when the power $r$ of Frobenius is sufficiently large, this dependence disappears and the geometric side of the trace formula reduces precisely to the Lefschetz number computed in Volume I. It then remains to compute the terms on the spectral side to achieve the desired goal of relating the Lefschetz numbers to the trace of Hecke operators. This is done in (the difficult) Chapter 11 of Volume II, at least for the terms attached to a so-called regular cuspidal datum. The result for general cuspidal data, due to Lafforgue, is stated without proof and is used to derive the main theorems in Chapter 12. The last chapter of Volume II discusses some conjectures related to the intersection cohomology of Drinfeld modular varieties. This volume also contains four very useful appendices on aspects of global harmonic analysis. For some reason, they are labelled D, E, F, G even though the fourth appendix of Volume I is also D.

As described above, these two volumes contain many results that are new and important. However, they are also the best source available for learning about the approach to zeta functions via the theory of automorphic representations. They contain a wealth of information, theorems, and calculations, laid before the reader in Laumon’s superb expository style. A wide range of topics from arithmetic geometry and local or global harmonic analysis are handled with ease and completeness. In short, these two volumes are a welcome addition to the literature on automorphic representations and are highly recommended.

Reviewed by Jonathan David Rogawski