Waltzing around Markov's Principle

for Wim Veldman with all best wishes
for a happy and productive future

Joan Rand Moschovakis
Occidental College (long retired) and MPLA

Nijmegen
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"Weak König's Lemma" WKL is König's Lemma KL for *detachable* subtrees of $2^{\mathbb{N}}$, constructively equivalent to

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

Adding a strong effective uniqueness hypothesis gives WKL!:

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0 \ \& $$
$$\forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\exists x \alpha(x) \neq \beta(x) \rightarrow \exists x [\rho(\overline{\alpha}(x)) \neq 0 \vee \rho(\overline{\beta}(x)) \neq 0]]$$
$$\rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

Theorem 1. (Ishihara, J. Berger, Schwichtenberg, all [2005]) (Using AC_{00}!) WKL! is constructively equivalent to Brouwer's fan theorem FT_d for a detachable bar, and hence to

$$\forall \alpha \in 2^{\mathbb{N}} \exists x \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists y \forall \alpha \in 2^{\mathbb{N}} \exists x \leq y \rho(\overline{\alpha}(x)) = 0.$$

Conclusion: WKL! is true intuitionistically.

Weakening the uniqueness hypothesis in WKL! gives WKL!!:

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0$$
$$\& \ \forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\forall x \rho(\overline{\alpha}(x)) = 0 \ \& \ \forall x \rho(\overline{\beta}(x)) = 0 \rightarrow \alpha = \beta]$$
$$\rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

Proposition 2. Constructively, WKL $\Rightarrow$ WKL!! $\Rightarrow$ WKL!.

Theorem 3. Constructively, WKL! $\not\Rightarrow$ WKL!! $\not\Rightarrow$ WKL.

Idea: Decompose WKL!! into Ishihara's intuitionistically dubious logical principle $\mathrm{MP}^{\vee}$:

$\neg\neg\exists x(\alpha(x) \neq 0 \vee \beta(x) \neq 0) \rightarrow \neg\neg\exists x \alpha(x) \neq 0 \vee \neg\neg\exists x \beta(x) \neq 0.$

and a mathematical principle $\neg\neg$WKL, following the example of Ishihara's decomposition [2005] of WKL and J. Berger's decomposition [2009] of WKL!. Establish the $\not\Rightarrow$s using realizability arguments (and the fact that $\mathrm{FT}_d \Leftrightarrow$ WKL! by Theorem 1).

In a little more detail: Recall Ishihara's [1993] decomposition of Markov's Principle into the conjunction of $\mathrm{MP}^{\vee}$ and WMP:

$$\forall\beta[\neg\forall n\beta(n) = 0 \vee \neg\forall n(\beta(n) = 0 \rightarrow \alpha(n) = 0)] \rightarrow \exists n\alpha(n) \neq 0,$$

where WMP is intuitionistically true by weak continuous choice.

Next establish constructively: WKL!! $\Leftrightarrow$ $\mathrm{MP}^{\vee}$ + $\neg\neg$ WKL.

Finally, to prove that WKL! $\not\Rightarrow$ WKL!! $\not\Rightarrow$ WKL, recall that FT_d $\Leftrightarrow$ WKL! and observe:

- FT_d and WKL!! are Kleene recursive function-realizable.
- FT_d is also ^{G}realizable (JRM [1971]), but $\mathrm{MP}^{\vee}$ is not; so WKL!! is not ^{G}realizable.
- WKL is not even Kleene recursive function-realizable, by Kleene's example of a recursive subtree of the binary tree which has (recursively) arbitrarily long finite branches but no recursive infinite branch.

Question: How much wilder an assumption is WKL!! than MP?

Heuristically, WKL!! says that a certain process which can have only one possible result – a binary sequence – will eventually produce that sequence, even though at any point in the process even the first element of that sequence may not be known with certainty. So WKL!! is like MP repeated countably many times.

Proposition 5.

(a) Over **M** + MP: $\mathrm{FT}_d \Leftrightarrow \neg\neg\mathrm{WKL} \Leftrightarrow \mathrm{WKL!!}$.

(b) Over **M** + WMP: $\mathrm{FT}_d + \mathrm{MP}^{\vee} \Leftrightarrow \mathrm{WKL!!}$.

(c) Over **M**: $\mathrm{FT}_d + \mathrm{MP} \Leftrightarrow \mathrm{WKL!!} + \mathrm{WMP}$.

(d) Over **FIM**: $\mathrm{WKL!!} \Leftrightarrow \mathrm{MP}^{\vee}$.

Conclusion: Intuitionistically, WKL!! is exactly as dubious as $\mathrm{MP}^{\vee}$, which is just as dubious as MP by Ishihara [1993].

Now consider the principle

(*): $\forall x \neg\neg \exists y \alpha(x,y) = 0 \to \neg\neg \forall x \exists y \alpha(x,y) = 0$

which is constructively equivalent to quantifier-free classical countable choice

qf-AC°_{00}: $\forall x \neg\neg \exists y \alpha(x,y) = 0 \to \neg\neg \exists \beta \forall x \alpha(x, \beta(y)) = 0.$

(*) is slightly less dubious than $MP^{\vee}$ because (like WMP) (*) is ^{G}realizable and hence consistent with **FIM** $+ \forall \alpha \neg\neg GR(\alpha)$.

The classical version FT°_{d} of FT_{d} is constructively equivalent to

$$\forall \alpha \in 2^{\mathbb{N}} \neg\neg \exists x \rho(\overline{\alpha}(x)) = 0 \to \neg\neg \exists y \forall \alpha \in 2^{\mathbb{N}} \exists x \leq y \rho(\overline{\alpha}(x)) = 0.$$

Proposition 6. (a) **FIM** + (*) does not prove MP or even $MP^{\vee}$.
(b) Constructively, FT_{d} + (*) $\Leftrightarrow FT^{\circ}_{d} \Leftrightarrow \neg\neg$WKL.

Conclusion: FT°_{d} is less dubious than $MP^{\vee}$ (hence less dubious than MP) from the intuitionistic viewpoint.

The general form of König's Lemma for the binary fan is KL:

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y R(\overline{\alpha}(x)) \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x R(\overline{\alpha}(x)).$$

KL!! is like KL but with the extra uniqueness hypothesis

$$\forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\forall x R(\overline{\alpha}(x)) \ \& \ \forall x R(\overline{\beta}(x)) \rightarrow \alpha = \beta].$$

Wim and others have studied intermediate versions of the fan theorem, weakening the detachability requirement of FT_d to a Π^0_1 monotone condition of one kind or another. Since König's Lemma is a classical contrapositive of an intuitionistically true fan theorem, it is tempting to consider the versions Σ^0_1-KL and Σ^0_1-KL!! of KL and KL!! when the predicate R is Σ^0_1. E.g., Σ^0_1-KL is

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \exists z \sigma(\overline{\alpha}(x), z) = 0 \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \exists z \sigma(\overline{\alpha}(x), z) = 0.$$

Proposition 7. Constructively, Σ^0_1-KL $\Rightarrow$ Σ^0_1-KL!! $\Rightarrow$ WKL.

Summary:

- Markov's Principle, which is classically consistent with Kleene and Vesley's intuitionistic analysis **FIM**, can be interpreted as saying that we have only the standard integers.
- (Ishihara) Constructively, $\mathrm{MP} \Leftrightarrow \mathrm{WMP} + \mathrm{MP}^{\vee}$ where **FIM** proves WMP but not $\mathrm{MP}^{\vee}$.
- (Berger, Ishihara, Schwichtenberg) $\mathrm{WKL!} \Leftrightarrow \mathrm{FT}_d$.
- **FIM** $+ \mathrm{MP}^{\vee}$ proves WKL!! but not WKL.
- MP entails (*), which is constructively equivalent to classical quantifier-free countable choice $\mathrm{qf\text{-}AC}^{\circ}_{00}$.
- **FIM** + (*) proves the classical form FT°_d of the detachable fan theorem, which is constructively equivalent to $\neg\neg$WKL, but does not prove MP.

Question: Can interesting mathematics be done in **FIM** + (*)?

Some references:

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