Now Under Construction: Intuitionistic Reverse Analysis

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Each variety of reverse analysis attempts to determine a minimal axiomatic basis for proving a particular mathematical theorem. Classical reverse analysis asks which set existence axioms are needed to prove particular theorems of classical second-order number theory. Informal constructive reverse analysis asks which constructive principles are needed to prove particular theorems of Bishop's constructive analysis, and which nonconstructive principles are equivalent over Bishop's constructive analysis to classical theorems. Intuitionistic reverse analysis asks which intuitionistically accepted properties of numbers and functions suffice to prove particular theorems of intuitionistic analysis using intuitionistic logic, and may also consider the relative strength of classical principles consistent with intuitionistic analysis.

This lecture sketches the current state of intuitionistic reverse analysis, in relation to its classical counterpart.

In a Nutshell: S. Simpson showed that many theorems of *classical analysis* are exactly provable in one of five subsystems of classical second-order number theory, distinguished by successively stronger *set existence* axioms. Intermediate systems are considered also.

Intuitionistic analysis depends on *function existence* principles: countable and dependent choice, fan and bar theorems, continuous choice. Intuitionistic logic distinguishes classically equivalent forms of countable choice. Many mathematical equivalents of the fan theorem have been identified. Building on a proof by T. Coquand, W. Veldman recently showed that over intuitionistic two-sorted recursive arithmetic **BIM** the principle of open induction on Cantor space is strictly intermediate between the fan and bar theorems, and is equivalent to intuitionistic versions of a number of classical theorems. R. Solovay proved that Markov's Principle is surprisingly strong in the presence of the bar theorem. This is work in progress.

Primitive recursive arithmetic $\mathbf{PRA}_0$ is a *quantifier-free* system based on intuitionistic logic with equality, in a language with variables over numbers, constants for $=$, 0, $'$ and all primitive recursive functions, and logical symbols $\&$, $\vee, \neg, \rightarrow, \leftrightarrow, (,)$. $\neg(0 = x')$ and the definitions of the function constants are axioms. A logical rule allows substitution of terms for variables, and quantifier-free mathematical induction is a rule. $x' = y' \rightarrow x = y$ and $x = y \vee \neg(x = y)$ are provable. $x < y$ abbreviates $x' \dot{-} y = 0$.

There is no difference between classical and intuitionistic $\mathbf{PRA}_0$.

If number quantifiers are added, mathematical induction is stated

$$A(0) \;\&\; \forall x(A(x) \rightarrow A(x')) \rightarrow A(x)$$

(for $A(x)$ quantifier-free) and the mathematical axioms are universally quantified, then the substitution rule becomes provable.

By removing the restriction on mathematical induction we get a definitional extension **PRA** of intuitionistic first-order arithmetic.

S. Simpson has organized **classical reverse mathematics**, distinguishing five main subsystems of classical second-order arithmetic $\mathbf{Z}_2$ extending a fragment of Peano arithmetic **PA** with restricted induction, in a language with variables and quantifiers over numbers and sets of numbers; individual constants 0, 1; operation constants $+$, $\cdot$ and predicate constants $=$, $<$, $\in$.

These subsystems have increasingly strong set existence axioms:

- $\mathbf{RCA}_0$ (recursive comprehension and Σ^0_1-induction),
- $\mathbf{WKL}_0$ ("weak König's Lemma"),
- $\mathbf{ACA}_0$ (arithmetical comprehension),
- $\mathbf{ATR}_0$ (arithmetical transfinite recursion),
- Π^1_1-$\mathbf{CA}_0$ (Π^1_1 comprehension).

(Note that $\mathbf{RCA}_0 + \mathbf{PRA}_0$ is a conservative extension of $\mathbf{RCA}_0$.)

More interesting for intuitionism are the versions with *unrestricted* mathematical induction: **RCA**, **WKL**, **ACA**, **ATR**, Π^1_1-**CA**.

An intuitionistic analogue of **RCA** can be defined in a two-sorted language with constants for a suitable list of primitive recursive functions and functionals (including pairing and coding of finite sequences), and variables and quantifiers over numbers and one-place number-theoretic functions. In **RCA** *every* set has a characteristic function so the difference in language is inessential for classical reverse mathematics. For *intuitionistic* reverse mathematics it is important because *only sets whose membership relation is effectively decidable* have characteristic functions.

Troelstra and Veldman treat *intuitionistic two-sorted recursive arithmetic* as an extension of **PRA**, but Kleene's original axiomatization of two-sorted intuitionistic arithmetic had only finitely many function constants, with the understanding that others could be added as needed. This approach seems more in keeping with the intuitionistic philosophy and we adopt it here.

Intuitionistic logic also affects the treatment of recursive comprehension. Classically, every Δ^0_1 function is recursive. For **RCA** and $\mathbf{RCA}_0$ the Δ^0_1-comprehension axiom takes the form

$$\forall x(\forall y A(x,y) \leftrightarrow \exists y B(x,y)) \rightarrow \exists X \forall x(x \in X \leftrightarrow \exists y B(x,y))$$

for $A(x,y)$, $B(x,y)$ quantifier-free. The intuitionistic analogue

$$\begin{aligned}\forall x[(\forall y \alpha(x,y) = 0 \leftrightarrow \exists y \beta(x,y) > 0)\\ \& \ (\forall y \beta(x,y) = 0 \rightarrow \exists y \alpha(x,y) > 0)]\\ \rightarrow \exists \gamma \forall x[\gamma(x) > 0 \leftrightarrow \exists y \alpha(x,y) > 0]\end{aligned}$$

is more complicated because Markov's Principle is not accepted intuitionistically. It is simpler to assume that the functions are closed under composition and primitive recursion and satisfy a quantifier-free form of the axiom of numerical choice:

$\mathbf{qf\text{-}AC}_{00}$: $\quad \forall x \exists y \alpha(x,y) = 0 \rightarrow \exists \beta \forall x \alpha(x, \beta(x)) = 0$

which assures the existence of every recursive total function.

Kleene's fragment $\mathbf{IA}_1$ of two-sorted intuitionistic arithmetic has variables $a, b, \ldots, x, y, z$ over numbers and $\alpha, \beta, \ldots$ over one-place number-theoretic functions; finitely many constants 0,',$+$,$\cdot$,exp,…,f_p for primitive recursive functions and functionals with their defining equations; full mathematical induction; λ-abstraction and λ-reduction. The equality axioms include $x = y \to \alpha(x) = \alpha(y)$, and equality between functions is defined extensionally by $\alpha = \beta \equiv \forall x \alpha(x) = \beta(x)$.

Kleene lets $2^x \cdot 3^y$ code the pair (x, y). The *length* $lh(x)$ of x is the number of nonzero exponents in its prime factorization, and $(x)_i$ is the exponent of p_i (with $p_0 = 2$). $Seq(x)$ abbreviates $\forall i < lh(x)\,((x)_i > 0)$, indicating that x codes a finite sequence; we write $\langle\,\rangle = 1$ and $\langle x_0, \ldots, x_n\rangle = \Pi_{i=0}^{n} p_i^{x_i+1}$. If $Seq(u)$ then $(u * \alpha)(n) = (u)_n \dot{-} 1$ if $n < lh(u)$, and $(u * \alpha)(lh(u) + n) = \alpha(n)$. $\overline{\alpha}(n) = \langle \alpha(0), \ldots, \alpha(n-1)\rangle$. If $k < lh(u)$ then $\overline{u}(k) = \overline{u * \lambda x.0}(k)$.

Kleene, who was interested in formalizing the full strength of intuitionistic analysis, originally assumed a strong axiom

$\mathbf{AC}_{01}$: $\quad \forall x \exists \alpha A(x, \alpha) \to \exists \beta \forall x A(x, \lambda y.\beta(x, y))$

of countable choice. To formalize recursive function theory he weakened AC_{01} to countable comprehension ("unique choice")

$\mathbf{AC}_{00}!$: $\quad \forall x \exists ! y A(x, y) \to \exists \alpha \forall x A(x, \alpha(x))$.

Comparing minimal formal systems for intuitionistic analysis, G. Vafeiadou observed that Kleene's $\mathbf{M}_1 = \mathbf{IA}_1 + \mathbf{AC}_{00}!$ proves

$\mathbf{CF}_0$: $\quad \forall x (A(x) \vee \neg A(x)) \to \exists \chi \forall x (\chi(x) = 0 \leftrightarrow A(x))$,

which is as strong, with classical logic, as full comprehension.

Theorem 1. (G. Vafeiadou) *Intuitionistic two-sorted recursive arithmetic* **IRA** can be expressed interchangeably by any one of:

- the subsystem $\mathbf{IA}_1^+ = \mathbf{IA}_1 + \text{qf-AC}_{00}$ of Kleene's $\mathbf{M}_1$ or
- A. S. Troelstra's *Elementary Analysis* **EL** or
- Wim Veldman's *Basic Intuitionistic Mathematics* **BIM**.

The interderivability of CF_0 with $\mathrm{AC}_{00}!$ over **IRA** affects the choice of stronger subsystems of **FIM**, where "A is a detachable subset of $\mathbb{N}$" is often expressed by $\forall x(A(x) \vee \neg A(x))$.

The natural intuitionistic analogue of **WKL** should be **IRA** plus some version of the Fan Theorem, and the natural intuitionistic analogue of **ATR** should be **IRA** plus some version of the Bar Theorem. But which version? Kleene gave *four* versions of his "bar theorem" axiom, all equivalent over $\mathbf{M}_1$ but not over **IRA**. The difference mattered when Solovay wanted to negatively interpret a classical system, with arithmetical countable choice and bar induction, in its intuitionistic counterpart.

W. Veldman, who deserves the lion's share of credit for developing intuitionistic reverse analysis, avoids the issue by working directly with characteristic functions, replacing $\forall x(A(x) \vee \neg A(x))$ by $\exists \zeta \forall x(\zeta(x) = 0 \leftrightarrow A(x))$. *In intuitionistic analysis, only detachable sets have characteristic functions*.

Brouwer's *binary fan* is the tree $2^{<\omega}$ of finite binary sequences, represented by their codes: $Bin(u) \equiv \forall i < lh(u)(1 \leq (u)_i \leq 2)$.

Proposition. (Veldman) **IRA** proves *detachable fan induction*:

Δ_0-**FI**: $\forall \alpha \in 2^{\mathbb{N}} \exists x \beta(\overline{\alpha}(x)) = 0 \;\&\; \forall u[Bin(u) \;\&$
$$\beta(u * \langle 0 \rangle) = 0 \;\&\; \beta(u * \langle 1 \rangle) = 0 \rightarrow \beta(u) = 0] \rightarrow \beta(\langle\,\rangle) = 0.$$

The *Detachable Fan Theorem* for $2^{\mathbb{N}}$ is

Δ_0-**FT**: $\forall \alpha \in 2^{\mathbb{N}} \exists x \beta(\overline{\alpha}(x)) = 0 \rightarrow \exists y \forall \alpha \in 2^{\mathbb{N}} \exists x \leq y \beta(\overline{\alpha}(x)) = 0.$

The *Enumerable Fan Theorem* Σ^0_1-**FT** is similar but with $\exists y \beta(\overline{\alpha}(x), y) = 0$ in place of $\beta(\overline{\alpha}(x)) = 0$.

Enumerable Bar Induction on $2^{\mathbb{N}}$, or Σ^0_1-*Fan Induction*, is

Σ^0_1-**FI**: $\forall \alpha \in 2^{\mathbb{N}} \exists x \exists y \beta(\overline{\alpha}(x), y) = 0 \;\&\; \forall u[Bin(u) \;\&$
$$\exists y \beta(u * \langle 0 \rangle, y) = 0 \;\&\; \exists y \beta(u * \langle 1 \rangle, y) = 0 \rightarrow \exists y \beta(u, y) = 0]$$
$$\rightarrow \exists y \beta(\langle\,\rangle, y) = 0.$$

Theorem 2. (H. Ishihara, W. Veldman) Over **IRA**:
$$\Sigma^0_1\text{-FT} \;\Leftrightarrow\; \Delta_0\text{-FT} \;\Leftrightarrow\; \Sigma^0_1\text{-FI}.$$

Weak König's Lemma for *detachable* subtrees of $2^{\mathbb{N}}$ is

WKL: $\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0$.

Adding a strong effective uniqueness hypothesis to WKL gives

WKL!: $\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0$ &

$$\forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\exists x \alpha(x) \neq \beta(x) \rightarrow \exists x [\rho(\overline{\alpha}(x)) \neq 0 \vee \rho(\overline{\beta}(x)) \neq 0]]$$
$$\rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

How "at most one" is expressed can be important intuitionistically. If the uniqueness hypothesis in WKL! is weakened to

$$\forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\forall x \rho(\overline{\alpha}(x)) = 0 \ \& \ \forall x \rho(\overline{\beta}(x)) = 0 \rightarrow \alpha = \beta]$$

a *stronger* Weak König's Lemma with uniqueness **WKL!!** results.

Theorem 3. (JRM)

1. Over **IRA**: WKL $\Rightarrow$ WKL!! $\Rightarrow$ WKL!, and none of the arrows can be reversed.
2. WKL!! is consistent with, but unprovable in, **FIM**.

The first modern formalization of Brouwer's results on the structure of the intuitionistic real numbers was by R. E. Vesley, in Kleene-Vesley [1965]. His careful work influenced all that followed.

Theorem 4. The following are equivalent over **IRA**:

(i) The Detachable Fan Theorem Δ_0-FT for $2^{\mathbb{N}}$.

(ii) Each pointwise continuous function on $[0, 1]$ with a modulus of continuity is uniformly continuous.

(iii) The Heine-Borel Theorem: Each enumerable open covering of $[0, 1]$ by intervals with rational endpoints has a finite subcover.

(iv) Brouwer's Approximate Fixed-Point Theorem for enumerable continuous functions on U $(= [0, 1] \times [0, 1])$.

(v) WKL!.

Proofs in recent literature are by J. Berger ((i) $\Leftrightarrow$ (ii)); H. Ishihara, I. Loeb, W. Veldman (independently) ((i) $\Leftrightarrow$ (iii)); W. Veldman ((i) $\Leftrightarrow$ (iv)); H. Ishihara and H. Schwichtenberg ((i) $\Leftrightarrow$ (v)).

Theorem 5. (essentially Kleene) Over **IRA** the Δ_0-FT for $2^{\mathbb{N}}$ is equivalent to the version for fans with bounded branching:

$$\mathbf{\Delta_0\text{-}FT}: \quad \forall\alpha[\forall x \alpha(x) \leq \beta(\overline{\alpha}(x)) \rightarrow \exists x \rho(\overline{\alpha}(x)) > 0] \\ \rightarrow \exists y \forall \alpha[\forall x \alpha(x) \leq \beta(\overline{\alpha}(x)) \rightarrow \exists x \leq y \rho(\overline{\alpha}(x)) > 0].$$

Simpson observed that classical **WKL** proves the corresponding generalization of Weak König's Lemma.

Since Δ_0-FT is interderivable with WKL over the classical version $\mathbf{IRA}^c = \mathbf{IRA} + (A \vee \neg A)$ of **IRA**, it seems reasonable to conclude

- **WFT** $=$ **IRA** $+$ Δ_0-FT *is an intuitionistic analogue of* **WKL**.

The next question is whether or not there is a natural intuitionistic analogue of **ACA**. Since Kleene showed that **FIM** proves

$$\neg\forall\alpha(\forall x \alpha(x) = 0 \vee \neg\forall x \alpha(x) = 0)$$

the answer will require some thought.

Brouwer proved the *Full Fan Theorem* using a stronger principle of bar induction on the *universal spread* $\mathbb{N}^{\mathbb{N}}$.

A *spread* is (determined by) a binary function σ satisfying

1. $\sigma(\langle\,\rangle) = 0$,
2. $\sigma(u) = 0 \leftrightarrow Seq(u)\ \&\ \exists x \sigma(u * \langle x \rangle) = 0$.

The *elements* of a spread are its infinite branches:

$$\alpha \in \sigma \equiv \forall x \sigma(\overline{\alpha}(x)) = 0.$$

A *fan* is a finitely branching spread, satisfying also

3. $\forall u[\sigma(u) = 0 \rightarrow \exists y \forall x(\sigma(u * \langle x \rangle) = 0 \rightarrow x \leq y)]$.

Let $Spr(\sigma) \equiv (1)\ \&\ (2)$ and $Fan(\sigma) \equiv (1)\ \&\ (2)\ \&\ (3)$.

While the bounded fan theorem is equivalent to Δ_0-FT over **IRA**, the literal statement of the detachable fan theorem is stronger:

Δ_0-$\mathbf{FT}_\sigma$: $\quad Fan(\sigma) \rightarrow$

$$[\forall \alpha \in \sigma \exists x \rho(\overline{\alpha}(x)) > 0 \rightarrow \exists y \forall \alpha \in \sigma \exists x \leq y \rho(\overline{\alpha}(x)) > 0].$$

A formula is *arithmetical* if it contains no function quantifiers. Parameters of both sorts are allowed.

Arithmetical countable comprehension ("unique choice") is

$\mathbf{AC}_{00}^{-}!$: $\forall x \exists ! y A(x,y) \rightarrow \exists ! \alpha \forall x A(x, \alpha(x))$

for arithmetical formulas $A(x,y)$.

Proposition. (Vafeiadou) Over **IRA** the following are equivalent:

(i) Arithmetical countable comprehension $\mathrm{AC}_{00}^{-}!$.

(ii) Every arithmetical relation $A(x)$ satisfying the law of excluded middle has a characteristic function:

$\mathbf{CF}_{0}^{-}$: $\forall x (A(x) \vee \neg A(x)) \rightarrow \exists \zeta \forall x (\zeta(x) = 0 \leftrightarrow A(x))$.

With classical logic, **IRA** $+ \mathrm{CF}_{0}^{-}$ gives full arithmetical comprehension. It seems reasonable to conclude that

- **IAC** $\equiv$ **WFT** $+ \mathrm{AC}_{00}^{-}!$ *is an intuitionistic analogue of* **ACA**.

To prove Δ_0-FT $\Rightarrow$ Δ_0-FT$_\sigma$ one needs e.g. Veldman's axiom

$\mathbf{W}$-Π^0_1-$\mathbf{AC}_{00}$: $\forall x \exists y \forall z \geq y\, \alpha(x,z) = 0 \rightarrow \exists \beta \forall x\, \alpha(x, \beta(x)) = 0$.

This is an instance of $\mathrm{AC}^-_{00}!$ because $\exists y \forall z \geq y \alpha(x,z) = 0 \leftrightarrow \exists! y[\forall z \geq y \alpha(x,z) = 0\ \&\ (y > 0 \rightarrow \alpha(x, y \dot{-} 1) > 0)]$.

Theorem 6. (essentially Kleene-Vesley) In **IAC** one can prove

1. Every finitely branching spread σ has bounded branching.
2. The literal version Δ_0-FT$_\sigma$ of the detachable fan theorem.
3. The fan theorem with a thin arithmetical bar A:

   $Fan(\sigma) \rightarrow [\forall \alpha \in \sigma \exists! x A(\overline{\alpha}(x)) \rightarrow \exists y \forall \alpha \in \sigma \exists! x \leq y A(\overline{\alpha}(x))]$.
4. Full bar induction on a fan with a thin arithmetical bar A:

   $Fan(\sigma) \rightarrow$

   $[\forall \alpha \in \sigma \exists! x A(\overline{\alpha}(x))\ \&\ \forall u[\sigma(u) = 0\ \&\ A(u) \rightarrow C(u)]\ \&$

   $\forall u[\forall y(\sigma(u * \langle y \rangle) = 0 \rightarrow C(u * \langle y \rangle)) \rightarrow C(u)] \rightarrow C(\langle\,\rangle)]$.

Each of the last three is equivalent to Δ_0-FT over **IRA** + $\mathrm{AC}^-_{00}!$.

Proof of Theorem 6.3 in **IAC**:

$Fan(\sigma) \to [\forall\alpha \in \sigma \exists! x A(\overline{\alpha}(x)) \to \exists y \forall\alpha \in \sigma \exists x \leq y A(\overline{\alpha}(x))]$.

Assume $Fan(\sigma)$ and $\forall\alpha \in \sigma \exists! x A(\overline{\alpha}(x))$. Then $\forall w[\sigma(w) = 0 \to A(w) \vee \neg A(w)]$, so $\exists\beta\forall w[\sigma(w) = 0 \to (\beta(w) = 0 \leftrightarrow A(w))]$. For such a β we have $\forall\alpha \in \sigma \exists! x \beta(\overline{\alpha}(x)) = 0$ so by Δ_0-FT: $\exists y \forall\alpha \in \sigma \exists x \leq y \beta(\overline{\alpha}(x)) = 0$, so $\exists y \forall\alpha \in \sigma \exists! x \leq y A(\overline{\alpha}(x))$.

Proof of Theorem 6.4 in **IAC**:

$Fan(\sigma) \to [\forall\alpha \in \sigma \exists! x A(\overline{\alpha}(x))\ \&\ \forall u[\sigma(u) = 0\ \&\ A(u) \to C(u)]\ \&$

$\forall u[\forall y(\sigma(u * \langle y \rangle) = 0 \to C(u * \langle y \rangle)) \to C(u)] \to C(\langle\,\rangle)]$.

Proof. Assume the hypotheses, so by Theorem 6.3 there is an n such that $\forall\alpha \in \sigma \exists! x \leq n A(\overline{\alpha}(x))$. If $\sigma(u) = 0$ and $lh(u) \leq n$ there are two cases: (i) $\exists x \leq lh(u) A(\overline{u}(x))$ or (ii) otherwise. We want to show that in each case $\exists x \leq lh(u) C(\overline{u}(x))$. Case (i) is no problem. If $\sigma(u) = 0\ \&\ \forall x \leq lh(u) \neg A(\overline{u}(x))$ then $lh(u) < n$, and we may choose u so that $n - lh(u)$ is as small as possible for case (ii). Then for all y with $\sigma(u * \langle y \rangle) = 0$: $A(u * \langle y \rangle)$, so $C(u * \langle y \rangle)$, so $C(u)$ after all. In particular, $C(\langle\,\rangle)$.

Three parameters determine the strength of a fan theorem and its corresponding fan induction principle:

(a) How is "finitely branching" expressed?

(b) What are the restrictions on the inductive predicate?

(c) What are the restrictions on the predicate defining the bar?

Intuitionistically there is more than one notion of finiteness.

Veldman calls a detachable set $A \subseteq \mathbb{N}$ *finite* if $\exists y \forall x (A(x) \rightarrow x \leq y)$; or *bounded-in-number* if there is a y such that A has no more than y elements. Formally, "A has no more than y elements" can be expressed in Π^0_1 form by

$$\forall u[lh(u) = y + 1 \ \& \ Inc(u) \rightarrow \exists i \leq y \neg A((u)_i)],$$

where $Inc(u) \equiv Seq(u) \ \& \ \forall i(i + 1 < lh(u) \rightarrow (u)_i < (u)_{i+1})$.

A set $A \subseteq \mathbb{N}$ is *almost finite* if $\forall \alpha[Inc(\alpha) \rightarrow \exists x \neg A(\alpha(x))]$, where $Inc(\alpha) \equiv \forall n(\alpha(n) < \alpha(n + 1))$.

An *approximate fan* is a spread σ such that if $\sigma(u) = 0$ then the set of all x such that $\sigma(u * \langle x \rangle) = 0$ is bounded-in-number.

Veldman's *Approximate Fan Theorem* is

$$\textbf{AFT}: \quad EAF(\sigma) \ \& \ \forall\alpha[\alpha \in \sigma \rightarrow \exists! x \rho(\overline{\alpha}(x)) > 0] \\ \rightarrow \forall\beta[Inc(\beta) \ \& \ \forall x \sigma(\beta(x)) = 0 \rightarrow \exists x \rho(\beta(x)) = 0],$$

where $EAF(\sigma)$ expresses: σ is an approximate fan, and there is a β such that for each x the set of all finite sequence codes u with $lh(u) = x \ \& \ \sigma(u) = 0$ has no more than $\beta(x)$ elements.

Theorem 7. (Veldman) Over **IRA**: AFT $\Rightarrow \Delta_0$-FT$_\sigma \Rightarrow \Delta_0$-FT. The first arrow cannot be reversed even over **IRA** + AC$_{01}$.

Veldman proved the last statement by deriving in **AFT** an intuitionistic version of the Ramsey Theorem, which entails the Paris-Harrington formula, which is not provable in **IRA** + AC$_{01}$ by a result of Goodman. And Troelstra proved that Δ_0-FT$_\sigma$ is conservative over **IRA** + AC$_{01}$ for arithmetical formulas.

Brouwer's *Bar Theorem* (really an axiom, as Kleene showed) is classically equivalent to transfinite recursion up to any countable ordinal. The principle of *Monotone* Σ^0_1-*Bar Induction* on $\mathbb{N}^{\mathbb{N}}$ is

$$\Sigma^0_1\text{-}\mathbf{BI}^m: \quad \forall\alpha\exists x\exists y\beta(\overline{\alpha}(x),y)=0$$
$$\& \ \forall u[Seq(u) \ \& \ \forall x\exists y\beta(u*\langle x\rangle,y)=0 \leftrightarrow \exists y\beta(u,y)=0]$$
$$\rightarrow \exists y\beta(\langle\,\rangle,y)=0.$$

The *Principle of Open Induction* on $\mathbb{N}^{\mathbb{N}}$ or $2^{\mathbb{N}}$ is

$$\mathbf{OI}: \quad \forall\beta[\forall\alpha(\alpha<\beta\rightarrow\exists y\rho(\overline{\alpha}(y))>0)\rightarrow\exists y\rho(\overline{\beta}(y))>0]$$
$$\rightarrow \forall\alpha\exists y\rho(\overline{\alpha}(y)>0$$

where $<$ is the lexicographic ordering on sequences.

Theorem 8. (T. Coquand) In **IRA** one can prove:

Σ^0_1-BI^m $\Rightarrow$ OI on $2^{\mathbb{N}}$ (and $[0,1]$) $\Rightarrow$ Heine-Borel for $[0,1]$.

The proofs of OI actually used bar induction on a subspread of $\mathbb{N}^{\mathbb{N}}$ with at most binary branching, an *approximate fan*!

Theorem 9. (Veldman) Over **IRA**:

$$\text{OI on } 2^{\mathbb{N}} \Leftrightarrow \text{OI on } [0,1] \Leftrightarrow \text{AFT}$$

Corollary. Open Induction on $2^{\mathbb{N}}$ (and $[0,1]$) is unprovable in **IAC** but can be proved in **IAC** $+ \Sigma^0_1$-BI^m.

Proof: By the proof of Theorem 7 with Troelstra's conservativity result (which holds for **IAC**); and by Theorems 8 and 9.

Veldman found other equivalents over **IRA** of Open Induction, including contrapositive versions of the Bolzano-Weierstrass and Monotone Convergence Theorems, and the following statement (whose converse is provable in **IRA**):

If $\forall\alpha\exists x\beta(\overline{\alpha}(x)) > 0$ then $\{u | Seq(u) \ \& \ \forall x < lh(u)\, \beta(\overline{u}(x)) = 0\}$ is well-founded under the Kleene-Brouwer ordering.

This suggests that perhaps

- **ABI** $=$ **IAC** $+ \Sigma^0_1$-BI^m *is an intuitionistic analogue of* **ATR**.

A Short Story. A few years ago R. Solovay wanted to prove that a classical system **S** with arithmetical comprehension and bar induction could be negatively interpreted in Kleene's neutral subsystem **B** of **FIM**. I pointed out that the negative interpretation of arithmetical comprehension is not accepted intuitionistically. Clearly Markov's Principle in the form

$$\mathbf{MP}_1: \quad \neg\neg\exists x\, \alpha(x) = 0 \rightarrow \exists x\, \alpha(x) = 0$$

together with Kleene's bar induction schema *in the form* ^{x}26.3b *of Kleene-Vesley*:

$$\mathbf{BI}_1: \quad \forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0\ \&\ \forall u[Seq(u)\ \&\ \rho(u) = 0 \rightarrow A(u)]$$
$$\&\ \forall u[Seq(u)\ \&\ \forall n A(u * \langle n\rangle) \rightarrow A(u)] \rightarrow A(\langle\,\rangle)$$

proves the negative interpretation of BI_1. (This was *not* obvious for the other forms ^{x}26.3a,c,d.)

Finally Solovay finessed the issue of arithmetical comprehension, and completed the proof of his theorem, by proving

Solovay's Lemma. In **IRA** + BI_1 + MP_1 one can prove

$$\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow \exists y T(x, x, \overline{\alpha}(y))]$$

with Kleene's primitive recursive T-predicate.

Theorem 10. (Solovay) **IRA** + BI_1 + MP_1 proves:

1. $\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow \exists y\alpha(x, y) = 0]$.
2. $\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow A(x, \alpha)]$ for $A(x, \alpha)$ arithmetical.
3. *Kuroda's Principle* (*arithmetical double negation elimination*):
   $\mathbf{DNS}_0^-$: $\forall x\neg\neg A(x) \rightarrow \neg\neg\forall x A(x)$ ($A(x)$ arithmetical).

Theorem 11. (JRM) **IRA** + BI_1 + MP_1 proves

1. The constructive arithmetical hierarchy is proper.
2. An intuitionistic version of Δ^1_1 comprehension:

$$\forall x[\neg\neg\exists\alpha\forall z\beta(x, \overline{\alpha}(z)) = 0 \leftrightarrow \forall\beta\exists z\gamma(x, \overline{\alpha}(z)) = 0]$$
$$\rightarrow \neg\neg\exists\delta\forall x[\delta(x) = 0 \leftrightarrow \forall\beta\exists z\gamma(x, \overline{\alpha}(z)) = 0].$$

Veldman's **EnDec?!** is a logical principle equivalent to AFT over **IRA**, hence derivable in **IRA** $+ \Sigma^0_1$-BI^m. Working from Solovay's result and my observation that EnDec?! follows in **IRA** $+ \mathrm{MP}_1$ from $\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow \exists y\alpha(x, y) = 0]$, Veldman proved

Theorem 12. (Veldman) Over **IRA** $+ \mathrm{MP}_1$:

EnDec?! $\Leftrightarrow$ $\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow \exists y\alpha(x, y) = 0]$ $\Leftrightarrow \Sigma^0_1$-$\mathrm{BI}^m$.

So with classical logic, Σ^0_1-BI^m would not add strength to **IAC**. Markov's Principle MP_1, although not accepted by Brouwer, is consistent with **FIM** relative to **B** $+ \mathrm{MP}_1$, as Kleene showed using function-realizability. F. Waaldijk and others have shown that MP_1 settles many questions in intuitionistic analysis and topology.

So the proposal for **ABI** $=$ **IAC** $+ \Sigma^0_1$-BI^m as an intuitionistic analogue of **ATR** must be considered tentative.

An intuitionistic analogue of Π^1_1-CA is needed to complete this proposed correspondence with Simpson's five subsystems.

So far we have not considered the axioms of countable choice, which were accepted by Brouwer and Kleene. Even to prove that every Cauchy sequence of rationals has a modulus of convergence, one seems to need full Π^0_1 countable choice:

Π^0_1-$\mathbf{AC}_{00}$: $\quad \forall x \exists y \forall z \alpha(x, y, z) = 0 \rightarrow \exists \beta \forall x \forall z \alpha(x, \beta(x), y) = 0$,

which does not follow from $\mathrm{AC}_{00}!$, as S. Weinstein showed in his PhD dissertation.

Proposition. Over **IRA** the following are equivalent:

1. Veldman's axiom W-Π^0_1-AC_{00}
2. Δ_0-$\mathbf{AC}^m_{00}$: $\quad \forall x \exists^m y\, \alpha(x, y) = 0 \rightarrow \exists \beta \forall x\, \alpha(x, \beta(x)) = 0$
   where $\exists^m y B(y) \equiv \exists y B(y)\ \&\ \forall y \forall z (B(y)\ \&\ y \leq z \rightarrow B(z))$
3. Π^0_1-$\mathrm{AC}_{00}!$.

Note that $\exists^m y B(y)$ is in general *stronger* than $\exists y \forall x \geq y B(y)$. This will be important in the sequel.

Kleene implicitly incorporated a classically correct choice principle

(*): $\forall\alpha\exists y R(\overline{\alpha}(y)) \rightarrow$

$$\exists\sigma\forall\alpha[\exists! x\sigma(\overline{\alpha}(x)) > 0 \ \& \ \forall x\forall y(\sigma(\overline{\alpha}(x)) = y + 1 \rightarrow R(\overline{\alpha}(y))],$$

into his version CC_{10} of Brouwer's classically false axiom of continuous choice. Troelstra's "neighborhood function principle" is another version of (*). Since *classical* bar induction conflicts with continuous choice, the conclusion cannot be sharpened to $\exists\sigma\forall\alpha[\exists! x\sigma(\overline{\alpha}(x)) > 0 \ \& \ \forall x(\sigma(\overline{\alpha}(x)) > 0 \rightarrow R(\overline{\alpha}(x)))]$.

However, *monotone* bar induction holds intuitionistically:

$\mathbf{BI}_{mon}$: $\forall\alpha\exists^m y R(\overline{\alpha}(y)) \ \& \ \forall u[Seq(u) \ \& \ R(u) \rightarrow A(u)]$

$$\& \ \forall u[Seq(u) \ \& \ \forall n A(u * \langle n \rangle) \rightarrow A(u)] \rightarrow A(\langle\,\rangle).$$

Vafeiadou and I have suggested a *monotone choice principle*

$\mathbf{AC}^m_{1/2,0}$: $\forall\alpha\exists^m y R(\overline{\alpha}(y)) \rightarrow$

$$\exists\sigma\forall\alpha[\exists! x\sigma(\overline{\alpha}(x)) = 0 \ \& \ \forall x(\sigma(\overline{\alpha}(x)) = 0 \rightarrow R(\overline{\alpha}(x))].$$

Proposition.

1. BI_{mon} is interderivable with BI_1 over **IAC** + $\mathrm{AC}^m_{1/2,0}$.
2. Σ^0_1-BI^m is interderivable with Σ^0_1-BI_1 in **IAC** + Σ^0_1-$\mathrm{AC}^m_{1/2,0}$.
3. In **IAC** + Π^0_1-AC^m_{00} one can prove that every Cauchy sequence of rationals has a modulus of convergence.
4. In **IAC** + Π^1_1-$\mathrm{AC}^m_{1/2,0}$ one can prove that every continuous function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ has a modulus of convergence and every Cauchy sequence of reals has a Cauchy modulus.

- Is **MBI** = **ABI** + Π^1_1-$\mathrm{AC}^m_{1/2,0}$ + Π^1_1-BI_1 a reasonable intuitionistic analogue of Π^1_1-**CA**?

The subsystem **IRA** + AC_{01} + BI_1 + $\mathrm{AC}^m_{1/2,0}$ of Kleene's **FIM** properly extends his neutral basic system **B**, and proves BI_{mon}, the full fan theorem FT and full fan induction FI.

- Does **IRA** + AC_{01} + BI_1 + $\mathrm{AC}^m_{1/2,0}$ correspond to $\mathbf{Z}_2$?

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and many more by W. Veldman, most available from arXiv.