Choice Sequences and Their Uses

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Brouwer's (destructive) "First Act of Intuitionism" questioned the universal applicability of the classical laws of double negation and excluded third. The resulting limitation to intuitionistic (constructive) reasoning made possible – and was justified by – Brouwer's "Second Act of Intuitionism" which accepted arbitrary choice sequences of natural numbers as legitimate mathematical objects, and required every function defined on all choice sequences to be continuous in the initial segment topology.

In the 20th century Heyting, Kleene, Vesley, Kreisel, Troelstra, and others clarified Brouwer's intuitionistic logic and mathematics by means of formal axiomatic systems; finally choice sequences could be compared with classical number-theoretic functions, and Brouwer's universal spread with classical Baire space. We explain this development, with the advantages of considering Brouwer's choice sequences as individual objects in the process of generation, spreads as structured sets, and species as extensional properties.

The "First Act of Intuitionism"

Brouwer (1908, "The unreliability of the logical principles") accepted the universal validity of $\neg\neg(A \vee \neg A)$ but not $A \vee \neg A$. He wrote: "Consequently the theorems which are usually considered as *proved* in mathematics, ought to be divided into those that are *true* and those that are *non-contradictory*."

Brouwer (1923, "Intuitionist splitting of the fundamental notions of mathematics") distinguished between the classical meaning of negation as falsity (justifying $\neg\neg A \rightarrow A$), and his interpretation of negation as absurdity (justifying only $\neg\neg\neg A \rightarrow \neg A$).

Heyting (1930) axiomatized *intuitionistic propositional and predicate logic* as subsystems of classical systems with axioms for $\&, \vee, \rightarrow, \neg, \forall$ and $\exists$. He characterized intuitionistic negation by $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$ and $(\neg A \rightarrow (A \rightarrow B))$, and gave a formal proof of $\neg\neg(A \vee \neg A)$ (whence $\forall x \neg\neg(A(x) \vee \neg A(x))$).

$\neg\neg\forall x(A(x) \vee \neg A(x))$ is unprovable (Kleene, Nelson 1945-7).

Brouwer's "First Act of Intuitionism" was not so much limiting as liberating. It made possible

- a faithful translation of classical arithmetic,
- subtle distinctions which are lost with classical logic, and
- divergent mathematical views.

Intuitionistic first-order arithmetic **IA**, with full mathematical induction, differs from classical (Peano) arithmetic **PA** only in its logic. If **IA** is consistent, so is **PA** (which contains it), since **PA** can be faithfully translated into **IA**. (Gödel 1933)

Classical $\forall$ and $\exists$ can be expressed intuitionistically by $\forall\neg\neg$ and $\neg\neg\exists$ respectively. (P. Kraus)

Markov's Principle MP: $\forall x(A(x) \vee \neg A(x))\ \&\ \neg\neg\exists x A(x) \rightarrow \exists x A(x)$ is independent of intuitionistic logic and arithmetic. (Kreisel 1959)

IA + MP + CT is consistent and proves $\neg\forall x(D(x,x) \vee \neg D(x,x))$, where $D(z,x)$ expresses $\{z\}(x)\downarrow$ and CT is "Church's Thesis" $\forall x \exists y A(x,y) \rightarrow \exists z \forall x[\{z\}(x)\downarrow \& A(x,\{z\}(x))]$. (Kleene-Nelson)

Brouwer's early treatment of the continuum

In his 1907 dissertation, Brouwer represented the rational numbers by pairs of integers with a decidable equivalence relation. Rational arithmetic is unchanged by intuitionistic logic.

Real numbers can be approximated arbitrarily closely by rationals. The constructive reals do not exhaust the continuum; they form a "scale" of the same order type η as the rationals.

Brouwer wrote "Mathematics can deal with no other matter than that which it has itself constructed;" and "*all* or *every* . . . tacitly involves the restriction: *insofar as belonging to a mathematical structure which is supposed to be constructed beforehand*."

Brouwer's primitive intuition of continuity or "fluidity" allowed him to complete a dense scale of order type η to the "measurable continuum," and "to state properties of the continuum as a 'matrix of points to be thought of as a whole'." He could quantify over the continuum, but were all its elements mathematical objects?

The “Second Act of Intuitionism”

For a more analytic, less geometrical-algebraic, treatment of the continuum, from (1918) on Brouwer developed the notions of *spread*, *choice sequence* and *species*. In “Historical background, principles and methods of intuitionism” (1952) he wrote:

The “Second Act of Intuitionism” explicitly recognizes “the possibility of generating new mathematical entities:

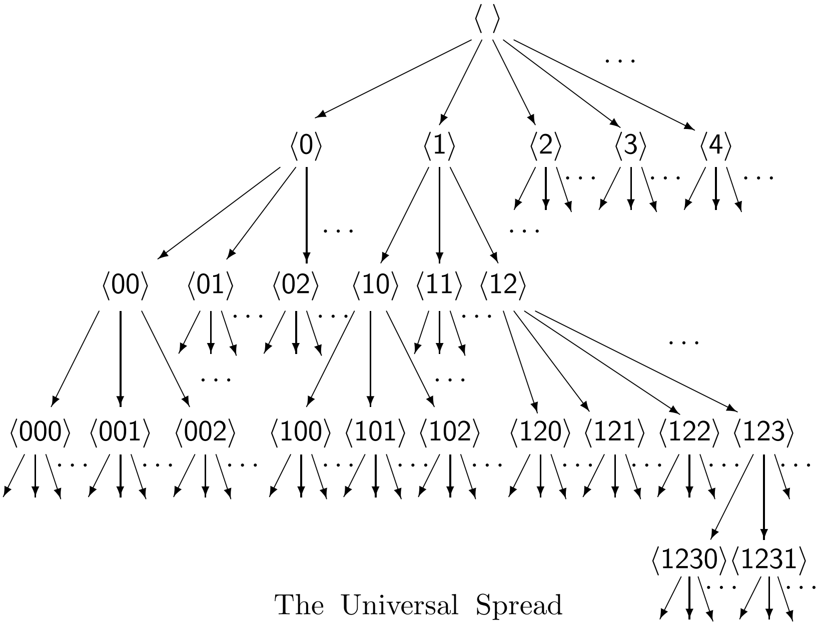
“firstly in the form of infinitely proceeding sequences $p_1, p_2, \ldots$ whose terms are chosen more or less freely from mathematical entities previously acquired ...;

“secondly in the form of mathematical species, i.e. properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it ...”

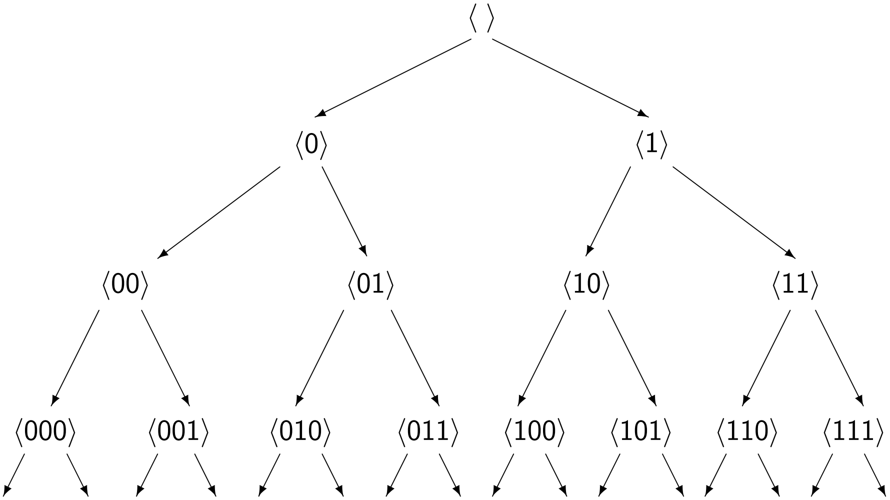
In "On the foundations of intuitionistic mathematics I" (1925) Brouwer describes the concept of structured set (later *spread*):

"A set [*menge*] is a law on the basis of which, if repeated choices of arbitrary natural numbers are made, each of these choices either generates a definite sign series ... or brings about the inhibition of the process ...; for every $n > 1$, after every ... uninhibited sequence of $n - 1$ choices, at least one natural number can be specified that, if selected as the nth number, does *not* bring about the inhibition of the process. Every sequence of sign series generated in this manner by an unlimited choice sequence (and hence generally not representable in a finished form) is called an *element of the set*. We shall also speak of the common mode of formation of the elements of a set M as ... *the set M*."

Brouwer reasoned that any process assigning a natural number $\nu(\alpha)$ to each element α of a spread must be *continuous* in the initial segment topology. His *universal spread* (no restrictions) and *binary fan* are intuitionistic versions of Baire and Cantor space.



The Universal Spread



The Binary Fan

Somewhat anachronistically, we may say that an infinitely proceeding sequence or "*choice sequence*" α is

- *free* or *arbitrary* as long as it is subject to no restrictions,
- *lawlike* (Brouwer) if all of its values are determined in advance according to some fixed law,
- *lawless* (Kreisel) if no more than a finite initial segment can ever be known in advance,
- *hesitant* (Troelstra) if it starts out free, but at any stage may be restricted to continue as lawlike,
- *relatively lawless* if every lawlike predictor correctly predicts some next values of α on the basis of values already chosen.

Cf. Kreisel's "Lawless sequences of natural numbers" (1967); Troelstra's "Choice Sequences" (1977); JRM (1986-1996, 2015).

Apparently Brouwer did not express an opinion on Church's Thesis but it seems reasonable that all recursive sequences are lawlike, and all lawlike sequences are definable in some sense.

After WWII Kleene was developing recursive function theory, on the way to proving normal form and hierarchy theorems for arithmetical and analytical relations. He defined a computational interpretation ("number realizability," 1945) to distinguish **IA** from **PA**, and David Nelson verified its correctness. Gene Rose (1952) found a propositional formula which was intuitionistically unprovable although all its arithmetical instances were realizable; so number realizability did not fully capture intuitionistic validity.

Kleene's "Introduction to Metamathematics" (1952) treated intuitionistic and classical logic and arithmetic in parallel, consistently distinguishing constructive from classical reasoning in mathematics and metamathematics. The continuum was omitted – the publisher imposed a strict page limit – but in 1950 Kleene visited Amsterdam and in 1952 Brouwer lectured in Canada and the US, visiting Kleene in Wisconsin. By 1957 Kleene had a function-realizability interpretation for part of intuitionistic analysis.

Formalizing intuitionistic analysis

Heyting's (1930) formalization of analysis was as obscure as his exposition of choice sequences and spreads in "Intuitionism: An Introduction" (1956) was lucid. Heyting lectured in the US in 1958 (I heard him in Berkeley). The formal system **FIM** of Kleene and Vesley (1965) and Kleene (1969) was guided by Heyting (1956).

FIM is a one-schema extension of a neutral subsystem **B** which has the same mathematical axioms as a classical theory **C** of numbers and number-theoretic functions, making comparisons easy. Competing formal systems by Kreisel and Troelstra (1970) and Myhill (1968, 1970) had variables over *lawlike* sequences.

Troelstra (1973) formalized Heyting's arithmetic of species **HAS**, but variables over intuitionistic species are not needed for intuitionistic analysis. *Detachable* species S of numbers satisfy $\forall n(n \in S \vee \neg n \in S)$ and so have characteristic functions, while properties of *definable* species can be expressed by schemas.

Kleene used *type-0* variables $\mathrm{m}, \mathrm{n}, \ldots, \mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{m}_1 \ldots$ over the natural numbers and *type-1 variables* $\alpha, \beta, \gamma, \ldots, \alpha_1, \ldots$ over infinitely proceeding sequences of numbers. *Prime formulas* are of the form $\mathrm{s} = \mathrm{t}$ where s, t are *terms* of type 0. Equality is extensional: $\alpha = \beta$ abbreviates $\forall \mathrm{x}(\alpha(\mathrm{x}) = \beta(\mathrm{x}))$.

IA$_1$ is *two-sorted intuitionistic arithmetic*, formalized using finitely many primitive recursive function constants, parentheses denoting function application, and Church's λ binding a type-0 variable.

The mathematical axioms of **IA**$_1$ are $(\mathrm{x} = \mathrm{y} \to \alpha(\mathrm{x}) = \alpha(\mathrm{y}))$, the defining equations for the function constants, the schema of mathematical induction for all formulas $\mathrm{A}(\mathrm{x})$, and the λ-reduction schema $(\lambda \mathrm{x}.\mathrm{u}(\mathrm{x}))\,(\mathrm{s}) = \mathrm{u}(\mathrm{s})$.

IA$_1$ has a classical model in which the sequence variables range over all primitive recursive functions. **IA**$_1$ proves that formulas with no quantifiers, or only bounded number quantifiers, are *decidable* (i.e., they satisfy the law of excluded middle).

Minimal Analysis $\mathbf{M} = \mathbf{IA}_1 + \mathsf{AC}_{00}!$ where $\mathsf{AC}_{00}!$ is

$$\forall x \exists ! y A(x, y) \to \exists \alpha \forall x A(x, \alpha(x)).$$

Here $\exists!$ denotes "there is exactly one," in the natural sense that $\exists ! y B(y)$ abbreviates $\exists y B(y)$ & $\forall x \forall y (B(x)$ & $B(y) \to x = y)$. **M** is strong enough to formalize the theory of recursive partial functions and functionals (Kleene 1969).

By intuitionistic logic with the decidability of number-theoretic equality, $\mathbf{M} \vdash \mathsf{AC}_{01}!$ where $\mathsf{AC}_{01}!$ is the comprehension schema

$$\forall x \exists ! \alpha A(x, \alpha) \to \exists \beta \forall x A(x, (\beta)^x)$$

where $(\beta)^x$ is $\lambda y \beta(\langle x, y \rangle)$ (the *x*th section of β).

With classical logic (but not intuitionistically by Weinstein 1979), $\mathsf{AC}_{00}!$ is equivalent to AC_{00}: $\forall x \exists y A(x, y) \to \exists \alpha \forall x A(x, \alpha(x))$.

Kleene's *Basic Analysis* $\mathbf{B} = \mathbf{IA}_1 + \mathrm{AC}_{01} + \mathrm{BI}_1$ where AC_{01} is a stronger *countable choice* principle:

$$\forall x \exists \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, (\beta)^x)$$

and BI_1 is the axiom schema of bar induction (w ranges over codes $\langle a_1, \ldots, a_k \rangle$ for finite sequences, $*$ denotes concatenation):

$$\forall \alpha \exists x \rho(\overline{\alpha}(x)) = 0 \ \& \ \forall w[\rho(w) = 0 \rightarrow A(w)] \\ \& \ \forall w[\forall s A(w * \langle s \rangle) \rightarrow A(w)] \rightarrow A(\langle\,\rangle).$$

Intuitionistic Analysis $\mathbf{FIM} = \mathbf{B} + \mathrm{CC}_1$, where CC_1 is Kleene's algorithmic version of Brouwer's continuous choice principle:

$$\forall \alpha \exists \beta A(\alpha, \beta) \rightarrow \exists \sigma \forall \alpha [\{\sigma\}[\alpha] \downarrow \ \& \ \forall \beta (\{\sigma\}[\alpha] = \beta \rightarrow A(\alpha, \beta))].$$

Classical Analysis $\mathbf{C} = \mathbf{B} + (A \vee \neg A)$.

"The Foundations of Intuitionistic Mathematics" (Kleene-Vesley 1965) was careful about credit. Kleene proved

1. **FIM** is consistent relative to **B**, by function realizability.
2. **FIM** proves $\neg\forall\alpha(\forall \mathrm{x}\alpha(\mathrm{x}) = 0 \vee \neg\forall \mathrm{x}\alpha(\mathrm{x}) = 0)$.
3. If A is arithmetical (contains only number variables) then $\mathrm{A} \vee \neg\mathrm{A}$ is consistent with **FIM** by a classical argument. So **FIM** + **PA** is consistent relative to **B** + **PA**.
4. Markov's Principle MP_1: $\neg\forall \mathrm{x}\neg\alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\alpha(\mathrm{x}) = 0$ is consistent with, but unprovable in, **FIM**. The proof used "special realizability," inspired by the "modified realizability" Kreisel used to prove MP independent of **HA**.

He also gave

5. a proof that the recursive sequences do not form a classical model for **B**, and
6. a counterexample in **FIM** to the classical (nonmonotone) version of the bar theorem.

In **FIM** Vesley gave a precise, comprehensive formal development of Brouwer's intuitionistic continuum, based on Heyting (1956), Brouwer's (1930) "Die Struktur des Kontinuums," etc., including

7. a definition of *canonical real number generators* as infinitely proceeding sequences α of (numerical codes for) dual fractions satisfying the Cauchy condition $\forall \mathrm{x}\, |2\alpha(\mathrm{x}) - \alpha(\mathrm{x})| \leq 1$,
8. a proof of Brouwer's uniform continuity theorem,
9. a proof that the continuum is " 'indivisible' ... by any predicate $C(\alpha)$ expressible by a formula $\mathrm{C}(\alpha)$ of the system,"
10. proofs that the continuum is "dense in itself," "everywhere dense" and "separable in itself," and that [0,1] is "freely connected" and negatively compact.

This careful formal treatment verified that Kleene and Vesley had captured Brouwer's mature theory of the continuum, but was so detailed it was considered hard to read.

Kleene's "Formalized Recursive Functionals and Formalized Realizability" (1969) was even harder to read. It established

11. If $\vdash_{\textbf{FIM}} \exists x A(x)$ where $\exists x A(x)$ is closed, then $\vdash_{\textbf{FIM}} A(\mathbf{n})$ for some numeral **n**.
12. If $\vdash_{\textbf{FIM}} \exists \alpha A(\alpha)$ where $\exists \alpha A(\alpha)$ is closed, then there is a gödel number e of a general recursive function φ such that $\vdash_{\textbf{B}} \forall x \{\mathbf{e}\}(x) \downarrow$ and $\vdash_{\textbf{FIM}} \forall \alpha (\forall x \{\mathbf{e}\}(x) \simeq \alpha(x) \rightarrow A(\alpha))$.

So intuitionistic analysis can only prove the existence of individual recursive sequences, while Brouwer's bar and fan theorems fail if all sequences are assumed to be recursive.

Bishop's "Foundations of Constructive Analysis" (1967) developed a neutral, informal theory of numbers and number-theoretic functions with roughly the principles of Kleene's $\mathbf{M} + \mathrm{AC}_{01}$. In the JSL review (1970) Myhill wrote "An important difference [from Brouwer] is that the notion of 'free choice sequence' is dropped and the only sequences used are lawlike."

Kreisel and Troelstra's "Formal systems for some branches of intuitionistic analysis" (1970) described a minimal system **EL** of "elementary analysis" like Kleene's **M** but with variables a, b, c, ... over *lawlike* rather than arbitrary choice sequences, with two-sorted intuitionistic arithmetic, full mathematical induction and a *function-existence axiom* **AC-NN!** like AC_{00}!:

$$\forall x \exists ! y A(x, y) \rightarrow \exists a \forall x A(x, a(x)).$$

Troelstra's "Metamathematical Investigations of Intuitionistic Arithmetic and Analysis" (1973) redefined **EL**, weakening AC-NN! by restricting it to *quantifier-free* formulas $A(x, y)$. This final version of **EL** can still prove the existence of every general recursive function, but unlike the earlier **EL** (or Kleene's **M**) it cannot prove that every decidable property of natural numbers has a characteristic function (Vafeiadou 2012). Veldman is currently investigating reverse intuitionistic mathematics over a minimal system **BIM**, like **EL** but with variables over choice sequences.

For bar induction Kreisel and Troelstra let e vary over the class K of "Brouwer-operations" (lawlike monotone neighborhood functions coding continuous functionals on the universal spread). The theory of *inductive definitions* **IDB** extended **EL** by the axioms for K, and $\mathbf{IDB_1} = \mathbf{IDB} + \text{AC-NF}$.

For comparison with **FIM**, they added choice sequence variables $\alpha, \beta, \ldots$, extending **IDB** to **ELC**, and formulated new principles:

A (analytic data): $\quad \mathrm{A}(\alpha) \to \exists \mathrm{e}(\exists \beta(\mathrm{e}|\beta = \alpha) \ \& \ \forall \beta \mathrm{A}(\mathrm{e}|\beta))$,

BC-C (bar-continuity): $\quad \forall \alpha \exists \beta \mathrm{A}(\alpha, \beta) \to \exists \mathrm{e} \forall \alpha \mathrm{A}(\alpha, \mathrm{e}|\alpha)$,
(like continuous choice CC_1 but with a lawlike modulus) and

BC-F!: $\quad \forall \alpha \exists ! \mathrm{b} \mathrm{A}(\alpha, \mathrm{b}) \to \exists \mathrm{e} \exists \mathrm{b} \forall \alpha \mathrm{A}(\alpha, (\mathrm{b})_{\mathrm{e}(\alpha)})$.

Troelstra proved that $\mathbf{CS} = \mathbf{ELC} + \mathrm{A} + \text{BC-C} + \text{BC-F!}$ is a conservative extension of $\mathbf{IDB_1}$ and a conservative extension of **FIM**, and that $\mathbf{CS} \vdash \forall \alpha \neg\neg \exists \mathrm{b}(\alpha = \mathrm{b})$.

Kreisel (1968, "Lawless sequences of natural numbers") proposed a different conservative extension **LS** of $\mathbf{IDB}_1$, with the variables $\alpha, \beta, \ldots$ ranging over *intensionally lawless* (rather than arbitrary choice) sequences. The new axioms were

LS1 (density): $\forall \mathrm{w} \exists \alpha \, (\overline{\alpha}(\mathrm{lh}(\mathrm{w})) = \mathrm{w})$,

LS2 (discreteness): $\forall \alpha \forall \beta \, (\alpha = \beta \vee \neg(\alpha = \beta))$,

LS3 (open data): $\mathrm{A}(\alpha) \to \exists \mathrm{n} \forall \beta \, (\overline{\beta}(\mathrm{n}) = \overline{\alpha}(\mathrm{n}) \to \mathrm{A}(\beta))$

and LS4 (lawlike continuous choice BC-F, like BC-F! without !). Troelstra observed that in LS3 and LS4 the α must be required to be distinct from any other free lawless variables.

Other notions of choice sequence are based on projections of lawless sequences; cf. Chapter 12 of Troelstra and van Dalen, "Constructivism in Mathematics" (1988), which shows how to eliminate choice sequence variables from **CS** and **LS**. The authors observe that $\mathbf{IDB}_1$ is compatible with CT but Kleene's **B** is not.

However, even **FIM** is compatible with "weak Church's Thesis"

WCT: $$\forall\alpha\neg\neg\exists\mathrm{e}\forall\mathrm{x}\,(\{\mathrm{e}\}(\mathrm{x})\simeq\alpha(\mathrm{x}))$$

by JRM (1971). While $\mathbf{IDB}_1$ requires all continuous functionals to have *lawlike* moduli, **B** does not. And LS2 is essentially intensional.

For an extensional alternative, I added lawlike sequence variables a, b,... to Kleene's language and a lawlike comprehension axiom $\mathrm{AC}_{00}^{R}!$ to **B**, then defined a choice sequence α to be *relatively lawless* – R-lawless where R is the class of lawlike sequences – if *each lawlike predictor correctly predicts* α *somewhere*. Formally:

$$\mathrm{RLS}(\alpha)\equiv\forall\mathrm{b}[\forall\mathrm{w}\,\mathrm{Seq}(\mathrm{b}(\mathrm{w}))\rightarrow\exists\mathrm{x}\,\alpha\in\overline{\alpha}(\mathrm{x})*\mathrm{b}(\overline{\alpha}(\mathrm{x}))].$$

As in Fourman's 1981 Brouwer Symposium talk, α and β are *independent* if their merge $[\alpha,\beta]$ is relatively lawless.

The formal system **RLS** has a *relative density* axiom in place of Kreisel's discreteness axiom, and the α in open data and lawlike continuous choice must satisfy an independence restriction.

A formula of the language is *restricted* if its choice sequence quantifiers vary over relatively independent R-lawless sequences, e.g. $\forall\alpha(\mathrm{RLS}([\alpha,\beta]) \to \mathrm{B}(\alpha,\beta))$, $\exists\alpha(\mathrm{RLS}([\alpha,\beta])\ \&\ \mathrm{B}(\alpha,\beta))$.

IRLS is **B** (extended to the new language) plus the axioms

$\mathsf{AC}^{R}_{00}!$ $\quad \forall \mathrm{x}\exists!\mathrm{y}\mathrm{A}(\mathrm{x},\mathrm{y}) \to \exists \mathrm{b}\forall \mathrm{x}\mathrm{A}(\mathrm{x},\mathrm{b}(\mathrm{x}))$

for $\mathrm{A}(\mathrm{x},\mathrm{y})$ restricted, with no choice sequence variables free.

RLS1: $\quad \forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \to \exists\alpha[\mathrm{RLS}(\alpha)\ \&\ \alpha \in \mathrm{w}])$.

RLS2: $\forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \to \forall\alpha[\mathrm{RLS}(\alpha) \to \exists\beta[\mathrm{RLS}([\alpha,\beta])\ \&\ \beta \in \mathrm{w}]])$.

RLS3: $\quad \forall\alpha[\mathrm{RLS}(\alpha) \to (\mathrm{A}(\alpha) \to$
$\exists \mathrm{w}(\mathrm{Seq}(\mathrm{w})\ \&\ \alpha \in \mathrm{w}\ \&\ \forall\beta[\mathrm{RLS}(\beta) \to (\beta \in \mathrm{w} \to \mathrm{A}(\beta))]))]$.

RLS4: $\quad \forall\alpha[\mathrm{RLS}(\alpha) \to \exists \mathrm{b}\mathrm{A}(\alpha,\mathrm{b})] \to$
$\exists \mathrm{e}\exists \mathrm{b}\forall\alpha[\mathrm{RLS}(\alpha) \to \exists \mathrm{n}(\mathrm{e}(\alpha) = \mathrm{n}\ \&\ \mathrm{A}(\alpha, \lambda \mathrm{x}.\mathrm{b}(\langle \mathrm{n},\mathrm{x}\rangle))))]$.

In RLS3,4 the $\mathrm{A}(\alpha)$ and $\mathrm{A}(\alpha,\mathrm{b})$ must be restricted, with no choice sequence variables free but α.

IRLS proves $\forall \mathrm{b} \exists \alpha (\mathrm{b} = \alpha)$ and $\forall \alpha (\mathrm{RLS}(\alpha) \to \neg \exists \mathrm{b} (\alpha = \mathrm{b}))$, and that the class of *R*-lawless sequences has nice closure properties. In contrast, **CS** proves $\forall \alpha \neg\neg \exists \mathrm{b} (\alpha = \mathrm{b})$, and in **LS** the class of lawless sequences has no nice closure properties.

In **IRLS** every restricted formula E without free choice sequence variables is equivalent to a formula $\tau(\mathrm{E})$ without choice sequence variables. A lawlike subsystem **IR** of **IRLS** formalizes Bishop's analysis, and **IR** + MP + CT formalizes Russian recursive analysis.

Troelstra (1997) observed that the lawlike sequence variables could range over the *classical* sequences, treated as completed objects. Formally, let **RLS** = **IRLS** + RLEM, where for $\mathrm{A}(\alpha)$ restricted and with no choice sequence variables but α free:

RLEM is $\qquad \forall \alpha [\mathrm{RLS}(\alpha) \to \mathrm{A}(\alpha) \vee \neg \mathrm{A}(\alpha)].$

In the language without choice sequence variables, **R** = **IR** + LEM formalizes classical analysis. Assuming a particular definably well-ordered subset of Baire space is countable, the common extension **FIRM** of **FIM** and **RLS** is consistent.

Why it is still useful to consider choice sequences as individuals:

- By varying the logic, the lawlike part of Brouwer's universal spread can be viewed as constructive, recursive or classical.
- Bishop's constructive sequences satisfy countable choice.
- The recursive sequences satisfy recursive countable choice but not the bar or fan theorem.
- The classical sequences satisfy the bar theorem and countable choice, but not continuous choice.
- Brouwer's choice sequences satisfy the bar theorem, countable choice and continuous choice.
- If the class R of lawlike sequences is countable "from the outside" then there is a class of R-lawless sequences which is disjoint from R and Baire comeager, with classical measure 0.
- The R-lawless sequences satisfy restricted open data and continuous choice, but not the restricted bar theorem.

Endnotes:

- If the class R of lawlike sequences is countable, the R-lawless sequences are all the generic sequences with respect to properties of finite sequences of natural numbers which are definable over $(\omega, R, \omega^\omega)$ by restricted R-formulas with parameters from ω, R.
- Dragalin (1974, in Russian), van Dalen (1978, "An interpretation of intuitionistic analysis") and Fourman (1982) all suggested modeling lawless by generic sequences. Lawlike predictors were my idea, but maybe not only mine.
- *R-lawless* and *random* are orthogonal concepts, since a random sequence of natural numbers satisfies some lawlike regularity properties (e.g. the percentage of even numbers in its nth initial segment should approach .50 as n increases) while an R-lawless sequence satisfies none.

Now a little reverse intuitionistic mathematics related to Veldman's work on Open Induction. A spread determined by a spread-law σ will be called *anchored* if $\sigma(w * \langle 0 \rangle) = 0$ whenever $\sigma(w) = 0$.

Observation. The principle of Open Induction on the binary fan is a consequence, in $\mathbf{IA}_1 + \text{qf-AC}_{00}$ (and equivalently in **BIM** or **EL**), of the (detachable) fan theorem for an anchored Σ^0_1 subfan of the binary fan, which is equivalent to bar induction on a spread with at most binary branching.

The proof is essentially Coquand's proof of Open Induction on the binary fan by bar induction on a spread with at most binary branching. The point of the observation is that by allowing more complicated fan "laws" – using the flexibility afforded by choice sequence variables – we can reduce some bar induction arguments to intuitionistically equivalent fan induction arguments.

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