Unavoidable Choice Sequences

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Kleene's formalization of intuitionistic analysis **FIM** (Kleene and Vesley [1965], as extended by Kleene [1969]) includes bar induction, countable and continuous choice, but cannot prove that the constructive arithmetical hierarchy is proper.

Veldman showed that in **FIM** the constructive analytical hierarchy collapses at Σ^1_2.

These are serious obstructions to interpreting the constructive content of classical analysis, just as the collapse of the arithmetical hierarchy at Σ^0_3 in $\mathbf{HA} + \mathrm{MP}_0 + \mathrm{ECT}_0$ limits the scope and effectiveness of recursive analysis.

Question: Can we do better by working within classical extensions of nonclassical theories, or within classically correct theories obeying e.g. Church's Rule or Brouwer's Rule?

We work in a two-sorted language $\mathcal{L}$ with variables over numbers and one-place number-theoretic functions (*choice sequences*). Our base theory **M** – the minimal theory used by Kleene [1969] to formalize the theory of recursive partial functionals, function realizability and q-realizability – extends Heyting arithmetic to the two-sorted language, with extensional equality for functions.

M includes defining axioms for finitely many primitive recursive function constants, a λ-reduction schema, and the function comprehension schema $\forall x \exists ! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$.

An *$\mathcal{L}$-theory* is a consistent axiomatic extension of **M** in the language $\mathcal{L}$ (possibly enriched by additional primitive recursive function constants). An $\mathcal{L}$-theory may be *intuitionistic, classical* or *intermediate* depending on its underlying logic.

The $\mathcal{L}$-theories **T** which have been proposed so far to express parts of constructive mathematics typically have one or more of the following properties:

An *explicit* $\mathcal{L}$-theory T provides explicit witnesses for existential theorems:

(a) If $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$ is closed and $\vdash_{\mathbf{T}} \exists \mathrm{x} \mathrm{A}(\mathrm{x})$ then $\vdash_{\mathbf{T}} \mathrm{A}(\mathbf{n})$ for some numeral **n**.

(b) If $\exists \alpha \mathrm{A}(\alpha)$ is closed and $\vdash_{\mathbf{T}} \exists \alpha \mathrm{A}(\alpha)$, then for some $\mathrm{B}(\alpha)$ with only α free:

$$\vdash_{\mathbf{T}} \forall \alpha [\mathrm{B}(\alpha) \rightarrow \mathrm{A}(\alpha)] \ \& \ \exists ! \alpha \mathrm{B}(\alpha).$$

A *Brouwerian* $\mathcal{L}$-theory **T** satisfies *Brouwer's Rule*:

"If $\vdash_{\mathbf{T}} \forall \alpha \exists \beta \mathrm{A}(\alpha, \beta)$ then

$$\vdash_{\mathbf{T}} \exists \sigma \forall \alpha [\forall \mathrm{x} \exists \mathrm{y} (\{\sigma\}[\alpha](\mathrm{x}) \simeq \mathrm{y}) \ \& \ \mathrm{A}(\alpha, \{\sigma\}[\alpha])]."$$

A *recursively acceptable* $\mathcal{L}$-theory **T** satisfies *Markov's Rule*:

"If $\vdash_{\mathbf{T}} \neg\neg\exists \mathrm{x} \mathrm{A}(\mathrm{x})$ & $\forall \mathrm{x}[\mathrm{A}(\mathrm{x}) \vee \neg \mathrm{A}(\mathrm{x})]$ then $\vdash_{\mathbf{T}} \exists \mathrm{x} \mathrm{A}(\mathrm{x})$"

and *Church's Rule*:

"If $\vdash_{\mathbf{T}} \exists \alpha \mathrm{A}(\alpha)$ with $\exists \alpha \mathrm{A}(\alpha)$ closed, then

$$\vdash_{\mathbf{T}} \exists \mathrm{e}[\forall \mathrm{x} \exists ! \mathrm{y} \mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{y}) \ \& \ \forall \alpha[\forall \mathrm{x} \forall \mathrm{y}[\mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{y}) \rightarrow \alpha(\mathrm{x}) = \mathrm{U}(\mathrm{y})] \rightarrow \mathrm{A}(\alpha)]]."$$

If **T** is both recursively acceptable and explicit, then **T** evidently satisfies the *Church-Kleene Rule*:

"If $\vdash_{\mathbf{T}} \exists \alpha \mathrm{A}(\alpha)$ where $\exists \alpha \mathrm{A}(\alpha)$ is closed, then for a suitable *e*:

$$\vdash_{\mathbf{T}} \exists \alpha[\forall \mathrm{x}(\alpha(\mathrm{x}) \simeq \{\mathbf{e}\}(\mathrm{x})) \ \& \ \mathrm{A}(\alpha)]."$$

No classical $\mathcal{L}$-theory has any of these properties (except, of course, closure under Markov's Rule).

FIM has all these properties. So do the $\mathcal{L}$-theory **FIM** + MP_1 and its (classically correct) $\mathcal{L}$-subtheory $\mathbf{T}_1 \equiv \mathbf{M} + \mathsf{BI}_1 + \mathsf{MP}_1$, which prove that the constructive arithmetical hierarchy is proper. Here BI_1 is the bar induction schema and MP_1 is

$$\forall\alpha(\neg\neg\exists \mathrm{x}\alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\alpha(\mathrm{x}) = 0).$$

In addition to "saving the constructive arithmetical hierarchy," $\mathbf{T}_1$ has "more classical sequences" than **FIM**, in the following sense. If **T** is an $\mathcal{L}$-theory and

$$\vdash_{\mathbf{T}} \neg\neg\exists\alpha \mathrm{A}(\alpha),$$

then we say *"a sequence α satisfying $\mathrm{A}(\alpha)$ is unavoidable over* **T***."*

Only recursive sequences are unavoidable over **FIM** (JRM [1971]) but the characteristic functions of all arithmetical relations (with or without sequence parameters), and of all classically Δ^1_1 relations, are unavoidable over **FIM** + MP_1 and over $\mathbf{T}_1$ (Solovay, JRM, in JRM [2003]).

Definition. **T_2** comes from **FIM** by adding

I. $\neg\neg\forall x[A(x) \vee \neg A(x)]$ for arithmetical $A(x)$ (parameters of both sorts allowed, but no sequence quantifiers).

II. "Only classically Σ^1_1 sequences are unavoidable":

$$\forall\alpha\neg\neg\exists e\forall x\forall y[\alpha(x) = y \leftrightarrow \neg\neg\exists\beta\forall z\neg T(e, x, y, \overline{\beta}(z))].$$

III. "Every Π^1_1 sequence is unavoidable":

$$\forall e[\forall x\neg\neg\exists y\forall\beta\exists z T(e, x, y, \overline{\beta}(z))\,\&$$
$$\forall x\forall y\forall u(\forall\beta\exists z T(e, x, y, \overline{\beta}(z))\,\&\,\forall\beta\exists z T(e, x, u, \overline{\beta}(z)) \rightarrow y = u) \rightarrow$$
$$\neg\neg\exists\alpha\forall x\forall y[\alpha(x) = y \leftrightarrow \forall\beta\exists z T(e, x, y, \overline{\beta}(z))]].$$

T_2 is consistent by a classical realizability interpretation (a modification of my old ^{G}realizability) satisfying first-order Peano arithmetic **PA** but not MP_1.

Definition. A sequence ε *agrees with* an $\mathcal{L}$-formula E as follows.

1. Every ε *agrees with* a prime formula P.
2. ε *agrees with* $\mathrm{A} \,\&\, \mathrm{B}$, if $(\varepsilon)_0$ *agrees with* A and $(\varepsilon)_1$ *agrees with* B.
3. ε *agrees with* $\mathrm{A} \vee \mathrm{B}$, if $(\varepsilon(0))_0 = 0$ implies that $(\varepsilon)_1$ *agrees with* A, while $(\varepsilon(0))_0 \neq 0$ implies that $(\varepsilon)_1$ *agrees with* B.
4. ε *agrees with* $\mathrm{A} \to \mathrm{B}$, if, whenever α *agrees with* A, $\{\varepsilon\}[\alpha]$ is defined and *agrees with* B.
5. ε *agrees with* $\neg\mathrm{A}$, if ε *agrees with* $\mathrm{A} \to 1 = 0$.
6. ε *agrees with* $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$, if $(\varepsilon)_1$ *agrees with* $\mathrm{A}(\mathrm{x})$.
7. ε *agrees with* $\forall \mathrm{x} \mathrm{A}(\mathrm{x})$, if, for each x, $\{\varepsilon\}[x]$ is completely defined and *agrees with* $\mathrm{A}(\mathrm{x})$.
8. ε *agrees with* $\exists \alpha \mathrm{A}(\alpha)$, if $\{(\varepsilon)_0\}$ is completely defined and $(\varepsilon)_1$ *agrees with* $\mathrm{A}(\alpha)$.
9. ε *agrees with* $\forall \alpha \mathrm{A}(\alpha)$, if, for each sequence α, $\{\varepsilon\}[\alpha]$ is completely defined and *agrees with* $\mathrm{A}(\alpha)$.

Definition. Let ε be a Δ^1_1 sequence and E a formula of $\mathcal{L}$ with at most Ψ free. Let $\mathit{\Psi}$ be numbers and Δ^1_1 sequences interpreting Ψ.

1. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ a prime formula P, if P is *true*-$\mathit{\Psi}$.
2. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ A & B, if $(\varepsilon)_0$ ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ A and $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ B.
3. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\mathrm{A} \vee \mathrm{B}$, if $(\varepsilon(0))_0 = 0 \Rightarrow (\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ A, and $(\varepsilon(0))_0 \neq 0 \Rightarrow (\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ B.
4. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\mathrm{A} \rightarrow \mathrm{B}$, if ε agrees with $\mathrm{A} \rightarrow \mathrm{B}$ and, whenever α ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ A, $\{\varepsilon\}[\alpha]$ (is defined and) ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ B.
5. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\neg\mathrm{A}$, if ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\mathrm{A} \rightarrow 1 = 0$.
6. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$, if $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}, (\varepsilon(0))_0$ $\mathrm{A}(\mathrm{x})$.
7. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\forall \mathrm{x} \mathrm{A}(\mathrm{x})$, if, for each x, $\{\varepsilon\}[x]$ is defined and ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}, x$ $\mathrm{A}(\mathrm{x})$.
8. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\exists \alpha \mathrm{A}(\alpha)$, if $\{(\varepsilon)_0\}$ is defined and $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}, \{(\varepsilon)_0\}$ $\mathrm{A}(\alpha)$.
9. ε ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}$ $\forall \alpha \mathrm{A}(\alpha)$, if ε agrees with $\forall \alpha \mathrm{A}(\alpha)$ and, for each Δ^1_1 sequence α, $\{\varepsilon\}[\alpha]$ is defined and ${}^{\Delta^1_1}$*realizes*-$\mathit{\Psi}, \alpha$ $\mathrm{A}(\alpha)$.

Definition. A closed formula E is ${}^{\Delta^1_1}$*realizable* if and only if some Δ^1_1 sequence ε ${}^{\Delta^1_1}$realizes E. An open formula is ${}^{\Delta^1_1}$*realizable* if and only if its universal closure is.

We need a number of lemmas, differing little from those for ${}^{\mathrm{G}}$realizability, e.g.

Lemma 4. For each formula E there is a primitive recursive sequence ε^{E} which agrees with E.

Lemma 7. Let E contain free only Ψ. Then E is ${}^{\Delta^1_1}$realizable if and only if there is a recursive partial functional $\varphi[\Psi, \gamma] \simeq \lambda t.\varphi(\Psi, \gamma, t)$ such that, for some Δ^1_1 sequence δ: $\varphi[\Psi, \delta]$ is completely defined and agrees with E for every choice of Ψ, and if every sequence in the list Ψ is Δ^1_1 then $\varphi[\Psi, \delta]$ ${}^{\Delta^1_1}$realizes-Ψ E.

The $\varphi[\Psi, \delta]$ given by Lemma 7 is called a ${}^{\Delta^1_1}$*realizer for* E.

Lemma 9. (a) For each arithmetical formula $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})$ with no free variables other than $\beta, x_1, \ldots, x_k$, and for each Δ^1_1 sequence β, there is a Δ^1_1 function ϑ_β of $t, x_1, \ldots, x_k$ such that if $\vartheta[x_1, \ldots, x_k] = \lambda t.\vartheta_\beta(t, x_1, \ldots, x_k)$ then for all $x_1, \ldots, x_k$:

(i) $\vartheta[x_1, \ldots, x_k]$ agrees with $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})$.

(ii) $\vartheta[x_1, \ldots, x_k]$ ${}^{\Delta^1_1}$realizes-$\beta, x_1, \ldots, x_k$ $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})$ if and only if, under the intended classical interpretation, $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})$ is true-$\beta, x_1, \ldots, x_k$.

(b) With the same conditions on $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})$ and β, there is a Δ^1_1 sequence ψ which ${}^{\Delta^1_1}$realizes-β $\forall \mathrm{x}_1 \ldots \forall \mathrm{x}_\mathrm{k}[\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k}) \vee \neg \mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})]$. In particular, if $\mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})$ is purely arithmetical, then $\mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k}) \vee \neg \mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{k})$ is ${}^{\Delta^1_1}$realizable.

Theorem. If $\Gamma \vdash_{\mathbf{T}_2} \mathrm{E}$ and the formulas Γ are ${}^{\Delta^1_1}$realizable, so is E.

Proof. For each axiom E with only Ψ free we give a ${}^{\Delta^1_1}$realizer $\varphi[\Psi, \delta]$. Then, assuming that a ${}^{\Delta^1_1}$realizer exists for each premise of a rule of inference, we give a ${}^{\Delta^1_1}$realizer for the conclusion.E.g.

$\varphi[\Psi] \simeq \varphi[\Psi, \lambda t.0] \simeq \Lambda\sigma\lambda t.0$ is a ${}^{\Delta^1_1}$realizer for an instance of (I) with only Ψ free, since Lemma 9(b) gives a ${}^{\Delta^1_1}$realizer for $\forall \mathrm{x}[\mathrm{A}(\mathrm{x}) \vee \neg \mathrm{A}(\mathrm{x})]$, and (I) is the double negation of this formula.

$\varphi \simeq \varphi[\delta] \simeq \varphi[\lambda t.0] \simeq \Lambda\alpha\Lambda\pi\lambda t.0$ is a ${}^{\Delta^1_1}$realizer for the axiom (II) asserting that every sequence is classically Σ^1_1. Agreement is obvious; and for each Δ^1_1 sequence α there exist numbers f and, by the Spector-Gandy Theorem, also e so that for all x, y:

$$\begin{aligned} \alpha(x) = y \quad &\Leftrightarrow \quad (\gamma)(Ez)T(f, x, y, \overline{\gamma}(z)) \\ &\Leftrightarrow \quad (E\beta \in \Delta^1_1)(z)\overline{T}(e, x, y, \overline{\beta}(z)). \end{aligned}$$

$\varphi \simeq \Lambda\sigma\Lambda\pi\lambda t.0$ ${}^{\Delta^1_1}$realizes axiom (III).

Corollary 1. $\mathbf{T}_2$ is consistent, in fact every closed theorem of $\mathbf{T}_2$ has a recursive ${}^{\Delta^1_1}$realizer.

Proof. In the proof of the theorem, the parameter δ used in defining a ${}^{\Delta^1_1}$realizer for an axiom of $\mathbf{T}_2$ can always be taken to be recursive, and this property is preserved by the rules of inference. $0 = 1$ is not ${}^{\Delta^1_1}$realizable so $\mathbf{T}_2$ is consistent.

Corollary 2. $\mathbf{T}_2$ is Brouwerian and does not prove MP_1.

Proof. $\mathbf{T}_2$ has Brouwer's continuous choice principle as an axiom schema. Vesley's Schema VS, which (proves Brouwer's creating subject counterexamples and) is ${}^{\Delta^1_1}$realizable, contradicts MP_1.

Corollary 3. $\mathbf{T}_3 = \mathbf{T}_2 + \mathbf{PA}$ is a Brouwerian $\mathcal{L}$-theory which is not recursively acceptable.

Proof. $\mathbf{T}_3$ is consistent by ${}^{\Delta^1_1}$realizability. $\mathbf{T}_3$ proves $\forall \mathrm{x} \exists ! \mathrm{y} [\mathrm{y} \leq 1 \,\&\, (\mathrm{y} = 0 \leftrightarrow \exists \mathrm{z} \mathrm{T}(\mathrm{x}, \mathrm{x}, \mathrm{z}))]$ and hence $\exists \alpha \neg \exists \mathrm{e} \forall \mathrm{x} \exists \mathrm{y} (\mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{y}) \,\&\, \mathrm{U}(\mathrm{y}) = \alpha(\mathrm{x}))$, so violates Church's Rule.

References

Kleene, S. C. and Vesley, R. E. [1965]: *The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions*, North-Holland, Amsterdam.

Kleene, S. C. [1969]: *Formalized Recursive Functions and Formalized Realizability*, Memoirs of the Amer. Math. Soc. **89**.

Moschovakis, J. R. [1971]: "Can there be no nonrecursive functions?," Jour. Symb. Logic **26** No. 2, pp. 309-315.

Moschovakis, J. R. [2003]: "Classical and constructive hierarchies in extended intuitionistic analysis," Jour. Symb. Logic **68** No. 3, pp. 1015-1043.

Vesley, R. E. [1970]: "A palatable substitute for Kripke's Schema," in *Intuitionism and Proof Theory*, North-Holland.