Let $M$ be the minimal two-sorted extension of Heyting Arithmetic, with full
induction in the extended language, which was used e.g. by Kleene [1] to formalize
the theory of recursive partial functions of type 2. In addition to the defining
equations for finitely many primitive recursive function constants, $M$ has the function
existence (or “non-choice”) axiom schema
\[ AC_\alpha : \forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)), \]
but no axiom of countable or dependent choice. Let $T$ be $M + B_1 + MP_1$, where
$B_1$ is Brouwer’s principle of bar induction in the form
\[ B_1 : \forall \alpha \exists x \rho(\alpha(x)) = 0 \land \forall x (\rho(\alpha(x)) = 0 \lor \forall \rho A(\alpha(x) * (\rho)) \rightarrow A(\alpha(x))) \rightarrow A(()) \]
and $MP_1$ is Markov’s Principle in the form
\[ MP_1 : \forall \alpha \forall x \neg \alpha(x) = 0 \rightarrow \exists \alpha(x) = 0. \]
Then $T$ proves:
(i) Every predicate $A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)$ without function quantifiers, indeed
every (classically or constructively) $\Delta^1_1$ predicate, is classically decidable with
respect to its number variables; that is.
\[ \neg \forall x_1 \ldots \forall x_n \forall \alpha_1 \ldots \forall \alpha_m [A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m) \lor \neg A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)]. \]
Hence $\neg \exists \exists \forall x_1 \ldots \forall x_n [\beta(\langle x_0, \ldots, x_n \rangle) = 1 \leftrightarrow A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)].$
(ii) Every $\Delta^0_1$ predicate has a recursive characteristic function, and the graph
of every recursive function is $\Delta^0_1$ (both classically and constructively).
(iii) The constructive arithmetical hierarchy (with or without function parameters)
is proper.

Result (i) for arithmetical predicates is due to Robert Solovay (personal
communication). A proof of Solovay’s result, and proofs of (ii), (iii), and (i) for classically
$\Delta^1_1$ predicates, appear in [4] along with other hierarchy results in consistent
extensions of intuitionistic analysis. Observe that in $T$, every constructively $\Delta^1_1$
predicate is also classically $\Delta^1_1$, since $MP_1$ implies
\[ [\exists \alpha \forall x R(\alpha(x), z) \leftrightarrow \forall \beta \exists \exists y Q(\beta(y), z)] \rightarrow \neg \exists \exists \exists \forall x R(\alpha(x), z) \leftrightarrow \forall \beta \exists \exists \exists y Q(\beta(y), z) \]
if $R(w, z)$ and $Q(v, z)$ are quantifier-free. Results (ii) and (iii) use Kleene’s normal
form theorem; as an example, we sketch the proof of (iii).

**Theorem** $T$ proves $\Pi^0_n \neq \Delta^0_{n+1} \neq \Sigma^0_{n+1}$ and $\Sigma^0_n \neq \Delta^0_{n+1} \neq \Pi^0_{n+1}$ for $n \in \omega$, so
the constructive arithmetical hierarchy (with or without function parameters) is proper.

**Proof.** Since $\Pi^0_0 = \Sigma^0_0 \neq \Delta^0_1$ by (ii), and $\Pi^0_n \cup \Sigma^0_n \subseteq \Delta^0_{n+1} = \Sigma^0_{n+1} \cap \Pi^0_{n+1}$, it
will suffice to show by induction on $n$ that $\Sigma^0_{n+1} \neq \Delta^0_{n+1}$ and $\Pi^0_{n+1} \neq \Delta^0_{n+1}$.

**Basis.** $n = 0$. Kleene’s normal form theorem, proved in $M$ (cf. [1]), gives enumerating predicates
\[ R_1(x, y, \alpha) \equiv \exists z T(x, y, \alpha(z)) \text{ and } P_1(x, y, \alpha) \equiv \forall z \neg T(x, y, \alpha(z)) \]
for $\Sigma_1^0(y, \alpha)$ and $\Pi_1^0(y, \alpha)$ respectively, where $T(x, y, w)$ is quantifier-free. $M$ proves

(*) $\forall a \forall x \forall y [\neg R_1(x, y, \alpha) \leftrightarrow \neg P_1(x, y, \alpha)]$.

so $T$ proves that $R_1(x, y, \alpha)$ is not $\Pi_1^0$ and $P_1(x, y, \alpha)$ is not $\Sigma_1^0$.

**Induction Step.** By the induction hypothesis with the normal form theorem, there are predicates

$R_{n+1}(x, y, \alpha) \equiv \exists zC(x, y, z, \alpha)$ and $P_{n+1}(x, y, \alpha) \equiv \forall zD(x, y, z, \alpha)$

which enumerate (provably in $M$) $\Sigma_{n+1}^0(y, \alpha)$ and $\Pi_{n+1}^0(y, \alpha)$ respectively, such that $T$ proves

[(*), $\forall a \forall x \forall y \forall z [\neg D(x, y, z, \alpha) \leftrightarrow \neg C(x, y, z, \alpha)]$.

Fix $\alpha$. By (i), $T$ proves

$\neg \exists \exists \exists \exists \forall \exists \forall \exists \forall [\zeta((x, y, z)) = 0 \leftrightarrow C(x, y, z, \alpha)] \land \eta((x, y, z)) = 0 \leftrightarrow D(x, y, z, \alpha)]$

so $\neg \forall \exists \forall \exists \forall \exists \forall [D(x, y, z, \alpha) \leftrightarrow \neg C(x, y, z, \alpha)]$ by (*), and hence

(**) $\forall a \forall x \forall y [\neg R_{n+1}(x, y, \alpha) \leftrightarrow \neg P_{n+1}(x, y, \alpha)]$.

Thus $R_{n+1}(x, y, \alpha)$ is not $\Pi_{n+1}^0$ and $P_{n+1}(x, y, \alpha)$ is not $\Sigma_{n+1}^0$.

By [3], Kleene and Vesley’s theory $\text{FIM}$ of intuitionistic analysis (a nonclassical extension of $M + \text{B}1$, including Brouwer’s principle of continuous choice, from which the countable axiom of choice follows) is consistent with $\forall \alpha \neg \neg GR(\alpha)$. Results (ii)-(iii) imply that the consistent extension $\text{FIM} + \text{MP}_1$ of $T$ proves $\neg \forall \alpha \neg \neg GR(\alpha)$. Both $T$ and $\text{FIM} + \text{MP}_1$, like other theories considered in [4], satisfy Kleene’s recursive instantiation rule: If $\exists \alpha B(\alpha)$ is a closed theorem of the theory, so is $\exists \alpha [GR(\alpha) \land B(\alpha)]$, where $GR(\alpha)$ expresses “$\alpha$ is recursive.” Thus Markov’s Principle increases the classical (but not the constructive) content of the intuitionistic continuum.

Kleene’s example in [2], of a recursive fan in which every recursive branch (but not every branch) is finite, shows that the recursive sequences are an inadequate basis for intuitionistic analysis. Markov’s Principle helps to explain this fact without implying the constructive existence of nonrecursive sequences. From this point of view, results (i)-(iii) could be considered reasonably strong evidence for Markov’s Principle.

**References**


