MARKOV'S PRINCIPLE, MARKOV'S RULE AND THE NOTION OF CONSTRUCTIVE PROOF

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1. Introduction

In [39] Georg Kreisel reflected at length on *Church's Thesis* CT, the principle proposed in 1936 by Alonzo Church ([9]) as a definition: "We now define the notion ...of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers." An equivalent version with "recursive function" replaced by "function computable by a Turing machine" was proposed the same year by Alan Turing. Yiannis Moschovakis ([55]) observes that CT "*refers essentially to the natural numbers*, and so its truth or falsity depends on what they are."

Of course its truth or falsity also depends on what it means for a calculation to be effective, and this question forces a distinction between the constructive and classical meanings of CT. This distinction in turn leads to *Markov's Principle* MP, which asserts (in its original version) that if a recursive algorithm cannot fail to converge then it converges. In its capacity as a principle of unbounded search, MP refers essentially to the natural numbers so its truth or falsity depends on what they are, and in particular on the fact that ω is the order type of the standard natural numbers.

Markov's Principle is not generally accepted by constructive mathematicians. Beeson ([2], p. 47) notes that "...even [Markov and the Russian constructivists] keep careful track of which theorems depend on it and which are proved without it." Bishop constructivists accept neither CT nor MP, although their work is consistent with both. Kreisel [35] proved that MP is not a theorem of intuitionistic arithmetic. Brouwer and traditional intuitionists reject an analytical version of MP, but their reasoning has more to do with the nature of the intuitionistic continuum than with what the natural numbers are.

Markov's Rule MR, in contrast, is admissible for most formal systems **T** based on intuitionistic logic.[1] In its simplest form the rule states that if **T** proves that a particular recursive algorithm cannot fail to converge, then **T** proves that it converges. Whether or not appropriate versions of the rule are always admissible for constructive theories evidently depends on what a constructive proof is.

Constructive mathematics is often described simply as "mathematics done with intuitionistic logic." While this oversimplification ignores the difference between constructive and classical answers to the question of what constitutes a legitimate mathematical object, it does express a fundamental aspect of constructive proof.

In [20] Arend Heyting quotes Kreisel ([34]) as saying "...the notion of constructive proof is vague." Heyting objects that

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In addition to M. A. Marfori and M. Petrolo, I am grateful to George Dyson for a copy of [13].

[1] H. Friedman [14] and Dragalin [12] independently discovered a simple method for proving the admissibility of Markov's Rule which applies uniformly to pure and applied intuitionistic systems.

> ...the notion of vagueness is vague in itself, what we need, is a precise notion of precision. As far as I know, the only notion of this sort is based on a formal system.

But even for a formal system with a recursive proof predicate, "...the difficulty reappears if we ask what it means that a given formula A is provable," for instance if $\neg\neg\exists p(p \vdash \mathrm{A})$ rather than $\exists p(p \vdash \mathrm{A})$, leading back to Markov's Principle, which Heyting did not accept.

More than twenty years after Brouwer rejected the law of excluded middle, Heyting formalized intuitionistic propositional logic and intuitionistic arithmetic. Pure intuitionistic predicate logic was isolated as a proper subsystem of classical predicate logic; cf. Kleene [27] where several equivalent formalizations are studied. Here we follow Kreisel [37] in adopting the acronym **HPC** (for "Heyting predicate calculus") for a formal system of intuitionistic first-order predicate logic.[2] The question of what constitutes a constructive proof *depends essentially on whether* **HPC** *is sound and complete* for its intended intuitionistic interpretation, and this question leads again to Markov's Principle, as Gödel and Kreisel observed.

Thus Markov's Principle, Markov's Rule and the common notion of constructive proof appear to be legitimate candidates for investigation in the spirit of Kreisel's *ideal of informal rigour* (cf. [39]). Following Heyting's implicit advice, we consider versions of MP and MR, and principles weaker than MP, in the context of formal systems for intuitionistic logic, arithmetic and analysis.

One especially interesting intermediate principle is known to be equivalent to the *weak* completeness of **HPC** for Beth semantics, as shown in detail by Dyson and Kreisel ([13]). In 1962 Kreisel [37] suggested that this principle "may be provable on the basis of as yet undiscovered axioms which hold for the intended interpretation ... *So the problem whether* **HPC** *is weakly complete is still open.*"

We show that the Gödel-Dyson-Kreisel principle, which entails the equivalence of the double negations of the binary fan theorem and weak König's Lemma, is provable in a weak common subsystem of intuitionistic and classical analysis from a stronger intermediate principle which refutes "weak Church's Thesis."[3] This leads to the conjecture that both principles are correct on the intended interpretation of intuitionistic analysis as a theory of numbers and (arbitrary) free choice sequences, and thus **HPC** is weakly complete for the intended interpretation.

2. A little history

In the mid twentieth century, first Kleene and then Kreisel developed research programs to axiomatize Brouwer's intuitionistic analysis and establish its consistency. At the Summer Institute for Mathematical Logic sponsored by the American Mathematical Society and held at Cornell University, July 1 to August 2, 1957, Kleene spoke on recursive functionals of higher finite types and sketched an extension of his number-realizability interpretation to intuitionistic analysis. Kreisel lectured on continuous functionals and contrasted Gödel's *Dialectica* interpretation with his own no-counterexample interpretation of Heyting arithmetic,

[2] Troelstra and van Dalen [64] prefer **IQC**, abbreviating "intuitionistic quantificational calculus," and reserve **IPC** for its propositional subsystem.

[3] While "Every sequence is recursive" is inconsistent with Brouwer's fan theorem, weak Church's Thesis (expressing "There can be no nonrecursive sequences") is consistent with Kleene's full system of intuitionistic analysis; cf. [30], [49].

which is discussed in Kahle's, Kanckos' and Kohlenbach's chapters in this volume.[4] One month later, at the International Colloquium "Constructivity in Mathematics" in Amsterdam, August 26 to 31, 1957, Kleene and Kreisel each proposed an interpretation of analysis using "countable" (Kleene, [28]) or continuous (Kreisel [36]) functionals of finite type.

In 1965 after many years of intense work Kleene and Vesley [30] provided a full formal system **I** for Brouwer's theory of natural numbers and arbitrary choice sequences, together with a function-realizability interpretation which established the consistency of **I** *with or without a strong form of Markov's principle*, relative to a classically correct subsystem. Inspired by Kreisel's [35], Kleene defined a modified function-realizability interpretation (which he called "special realizability") to prove Markov's Principle independent of **I**.[5] He relativized his recursive function-realizability to show e.g. that **I** is consistent with first-order Peano arithmetic, and he and Vesley developed within **I** a significant part of Brouwer's theory of real numbers.

In August 1968 a seminal Conference on Intuitionism and Proof Theory was held at the State University of New York in Buffalo. The resulting volume [25] included important contributions by Kreisel, Bishop, Heyting, Myhill, Dana Scott, Feferman, and logicians of the next generation: Vesley, Dick de Jongh, Martin-Löf, William Howard, Dirk van Dalen, Anne Troelstra and others. Kleene's paper on formalized recursive functionals became [29]. Takeuti's course on proof theory and Troelstra's course on intuitionism were also published separately.

Kreisel's notion of "lawless" sequence appeared in [38] and was later improved by Heyting's student Troelstra, who spent most of a year with Kreisel after completing his doctoral dissertation in Amsterdam. They collaborated on [40] (see also [61]), an extended study of alternative formal systems for intuitionistic analysis emphasizing the role of lawlike and lawless sequences. After returning to Amsterdam Troelstra edited (and mostly wrote) an influential, comprehensive volume [62] on the metamathematics of intuitionistic arithmetic and analysis. He and Dirk van Dalen in Utrecht eventually coauthored [64], and produced generations of students with a solid knowledge of intuitionism from a neutral or classical viewpoint.

Most of Kleene's students worked in other areas, although many were influenced by his preference for constructive arguments.[6] Clifford Spector, probably his most brilliant student, died prematurely after extending Brouwer's bar theorem to prove the consistency of classical analysis ([60], completed for publication by Kreisel). After formalizing his function-realizability interpretations in [29] in order to establish that his formal systems satisfied a precise analogue of Church's Rule,[7] Kleene wrote relatively little on intuitionism.

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[4] At the same summer institute Martin Davis lectured on computable functionals of arbitrary finite type and Kreisel presented the Kreisel-Lacombe-Shoenfield Theorem (cf. [19]).

[5] On one occasion in the summer of 1963 Kleene took me with him to Kreisel's office at Stanford, where I silently witnessed a long, intense discussion – as I remember, rather one-sided. By that time Kleene and Vesley's manuscript, based on a formal system with choice sequence variables, was nearly complete.

[6] Anyone who has studied logic from [27] must have noticed Kleene's use of the superscript $^\circ$ to distinguish theorems whose proofs require classical reasoning from those, such as Gödel's first and second incompleteness theorems, which hold both classically and constructively.

[7] In particular, every closed existential theorem of **I** with or without Markov's Principle can be improved to provide a specific number or general recursive sequence as a witness.

3. MP and MR in the context of formal systems

In 1958 Kreisel ([35]) developed a modification of Kleene's number-realizability interpretation to show that the primitive recursive form of Markov's Principle, which is (classically) consistent with Heyting arithmetic **HA** by Kleene's number-realizability, is independent of **HA**. Since then logicians have analyzed versions of MP and MR over intuitionistic predicate calculus and over constructive and intuitionistic theories of natural numbers, real numbers, infinite sequences of natural numbers, and primitive recursive functions of all finite types. While a complete annotated bibliography of these investigations is beyond the scope of this article, in this and following sections we survey the folklore and contributions by many researchers over the past sixty years. Much more information is contained in the references and their bibliographies.

3.1. **Formal versions of MP.** Over intuitionistic arithmetic **HA** Markov's Principle can be rendered by a formula:

$$\mathrm{MP_0}: \quad \forall \mathrm{e}\forall \mathrm{x}[\neg\forall \mathrm{y}\neg \mathrm{T(e,x,y)} \rightarrow \exists \mathrm{y T(e,x,y)}]$$

where $\mathrm{T(e,x,y)}$ is a prime formula expressing Kleene's T-predicate ("y is a gödel number of a successful computation of $\{e\}(x)$"). Alternatively, it can be expressed by a schema:

$$\mathrm{MP_{QF}}: \quad \neg\forall \mathrm{x}\neg \mathrm{A(x)} \rightarrow \exists \mathrm{x A(x)},$$

where $\mathrm{A(x)}$ may contain additional free variables but must be "quantifier-free" (bounded quantifiers are allowed), or

$$\mathrm{MP_{PR}}: \quad \neg\forall \mathrm{n}\neg \mathrm{A(n)} \rightarrow \exists \mathrm{n A(n)}$$

where $\mathrm{A(n)}$ must express a primitive recursive relation of n and its other free variables. Over **HA** all three of $\mathrm{MP_0}$, $\mathrm{MP_{QF}}$ and $\mathrm{MP_{PR}}$ are equivalent.

The most general schematic version of Markov's Principle is

$$\mathrm{MP_D}: \quad \forall \mathrm{x}(\mathrm{A(x)} \vee \neg \mathrm{A(x)})\ \&\ \neg\forall \mathrm{x}\neg \mathrm{A(x)} \rightarrow \exists \mathrm{x A(x)},$$

(cf. [2], p. 47). Evidently **HA** + $\mathrm{MP_D} \vdash \mathrm{MP_{QF}}$ since if $\mathrm{A(x)}$ has only bounded quantifiers then **HA** $\vdash \forall \mathrm{x}(\mathrm{A(x)} \vee \neg \mathrm{A(x)})$. However, Smorynski ([59], p. 365) proved that **HA** + $\mathrm{MP_{QF}} \nvdash \mathrm{MP_D}$.[8] *A fortiori* $\mathrm{MP_D}$ is not provable in **HPC** from instances of $(\neg\forall \mathrm{x}\neg \mathrm{P(x)} \rightarrow \exists \mathrm{x P(x)})$ with $\mathrm{P(x)}$ prime. We leave the proof that **HPC** + $\mathrm{MP_D} \nvdash (\neg\forall \mathrm{x}\neg \mathrm{P(x)} \rightarrow \exists \mathrm{x P(x)})$ as an exercise for the reader.[9]

As principles of pure predicate logic, $\mathrm{MP_D}$ and variants can have no convincing constructive justification because the connection with the natural numbers is lost. Closed instances are *persistently consistent* with **HPC** in the sense that their double negations are provable, but the schemas are not persistently consistent because **HPC** $\nvdash \neg\neg\forall \mathrm{y}[\forall \mathrm{x}(\mathrm{P(x,y)} \vee \neg \mathrm{P(x,y)})\ \&\ \neg\forall \mathrm{x}\neg \mathrm{P(x,y)} \rightarrow \exists \mathrm{x P(x,y)}]$.[10]

Over a two-sorted theory such as Kleene's **I** ([30]) and subsystems, Troelstra's **EL** ([62]) or Veldman's **BIM** ([68]), Markov's Principle can be strengthened to

$$\mathrm{MP_1}: \quad \forall \alpha(\neg\forall \mathrm{x}\neg\alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\alpha(\mathrm{x}) = 0).$$

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[8]In the presence of Church's Thesis, $\mathrm{MP_{QF}}$ and $\mathrm{MP_D}$ are arithmetically equivalent. If CT! is the recursive comprehension principle $\forall \mathrm{x}\exists!\mathrm{y A(x,y)} \rightarrow \exists \mathrm{e}\forall \mathrm{x}\exists \mathrm{y}[\mathrm{T(e,x,y)}\ \&\ \mathrm{A(x,U(y))}]$ (where $\exists!\mathrm{y B(y)}$ abbreviates $\exists \mathrm{y B(y)}\ \&\ \forall \mathrm{z}(\mathrm{B(y)}\ \&\ \mathrm{B(z)} \rightarrow \mathrm{y} = \mathrm{z})$), then **HA** + CT! + $\mathrm{MP_{QF}} \vdash \mathrm{MP_D}$.

[9]Consider a two-node Kripke model with a constant domain of one element which satisfies $\mathrm{P(x)}$ at the leaf but not at the root.

[10]For a counterexample, consider a linear Kripke model with root 0, nodes $n \in \omega$ with $n < n+1$, $D(n) = \{\mathbf{0}, \ldots, \mathbf{n}\}$ and $\mathrm{P}(\mathbf{n+1}, \mathbf{n})$ true at all $k \geq n+1$.

As long as the characteristic function of Kleene's T predicate is adequately represented in the system, MP_1 entails MP_0, and if the axiom schema of countable (unique) choice for quantifier-free relations is present then MP_1 entails MP_{QF}. Whether or not MP_1 entails MP_D depends on whether or not the system proves that every decidable relation has a characteristic function; cf. [66], [65], [54]. The type-0 variable x is always intended to range over the constructive (standard) natural numbers, but the type-1 variable α may be interpreted as ranging over all infinite sequences of natural numbers or over a proper subset (or subspecies) of them; when α ranges over the recursive sequences, MP_1 expresses Markov's original intention.

3.2. **Markov's Rule and constructive arithmetical truth.** By the (uniform, syntactic) Friedman-Dragalin translation, every formal system mentioned so far has the property that if $\forall x(A(x) \vee \neg A(x))$ & $\neg\forall x\neg A(x)$ is provable so is $\exists x A(x)$; so Markov's Rule MR_D is admissible for these (and most other) formal systems based on intuitionistic logic. In [41] Leivant proved that MR_D holds for a system **HA*** (**HA** extended with transfinite induction over all recursive well-orderings) which he suggests may capture the notion of constructive arithmetical truth.

Any attempt to identify constructive arithmetical truth with provability in a consistent recursively enumerable extension of **HA** will of course be frustrated by Gödel's incompleteness theorem. However, over **HA** or **I** the conclusion of MR_{QF} for a closed formula $\exists x A(x)$ can be strengthened to "$A(\mathbf{n})$ is provable for some numeral **n**" because these systems satisfy *numerical existential instantiation*: If $\vdash \exists x E(x)$ where $\exists x E(x)$ is closed, then $\vdash E(\mathbf{n})$ for some numeral **n** ([26], [57], [29]). In this sense intuitionistic arithmetic and analysis succeed in expressing the constructive understanding that *we have only the standard natural numbers.*

Adding MP_{QF} to intuitionistic arithmetic or analysis preserves numerical existential instantiation for Σ^0_1 formulas and does not increase the stock of provably recursive functions. Peano arithmetic is Π^0_2-conservative over **HA** by the Gödel-Gentzen negative translation with MR_{QF}, and **I** + MP_{QF} is Π^0_2-conservative over **I** by [48]. These appear to be strong arguments for the constructive truth of the arithmetical forms of Markov's Principle.

3.3. **A note on metamathematical methods.** In his case study on informal rigour [39], Kreisel considers the possibility that there may be "simple conditions, easily verified for current intuitionistic systems, that imply easily the consistency of CT and closure under Church's rule." If so, the need for "detailed studies like ingenious realizability interpretations" would be eliminated.

The Friedman-Dragalin translation is a simple tool to prove intuitionistic formal systems are closed under Markov's rule. Coquand and Hoffmann [10] showed how the method can be extended to prove that MP_{QF} is Π^0_2-conservative over many intuitionistic systems. Kripke models of **HPC** and **HA**, and topological and Beth models of theories of choice sequences, help to establish the relative consistency and independence of MP and related principles.

But for delicate questions about the strength of variants and weakenings of Markov's principle over constructive and intuitionistic analysis, realizability interpretations (Kleene and Nelson's number-realizability, Kreisel's modified number-realizability, Kleene's function-realizability and $_{S}$realizability [30], ^{G}realizability [49], Lifschitz realizability ...) are among the best tools available. Some of these methods can be replaced by categorical ones (cf. [67]), which we do not use here.

4. Markov's principle and completeness properties of HPC

4.1. **Markov's principle and strong completeness of HPC.** A variant of $\mathrm{MP_{PR}}$ is intuitionistically equivalent to the strong completeness of **HPC** for the topological and Beth interpretations of intuitionistic logic. This fact is due to Gödel and Kreisel ([33]). In order to state it precisely, let β range over infinite (free choice) sequences of natural numbers and let $\mathrm{B}(\beta)$ abbreviate $\forall \mathrm{n}(\beta(\mathrm{n}) \leq 1)$. Then the following statements are equivalent:

(1) $\forall\beta_{\mathrm{B}(\beta)}\neg\forall \mathrm{n}\neg \mathrm{A}(\mathrm{n},\beta) \to \forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{n}\mathrm{A}(\mathrm{n},\beta)$ holds for all primitive recursive relations $\mathrm{A}(\mathrm{n},\beta)$ between natural numbers and free choice sequences.

(2) **HPC** is complete for its topological interpretation and for Beth's semantic construction of intuitionistic logic [4], in the sense that every formula of the predicate language which is valid under either interpretation is provable in **HPC**.

Evidently $\mathrm{MP_{PR}}$ (with a choice sequence variable β free) entails (1), and (1) (with a choice sequence variable α free) entails $\mathrm{MP_1}$ by the following argument. Let $\mathrm{A}(\alpha,\mathrm{n},\beta)$ abbreviate $\exists \mathrm{m} < \overline{\beta}(\mathrm{n})(\alpha(\mathrm{m}) = 0)$, where $\overline{\beta}(\mathrm{n})$ abbreviates $\Pi_{\mathrm{m}<\mathrm{n}}\mathrm{p}_{\mathrm{m}}^{\beta(\mathrm{m})+1}$ and $\mathrm{p_m}$ represents the mth prime number. Assume (a) $\neg\forall \mathrm{x}\neg\alpha(\mathrm{x}) = 0$. Then also (b) $\forall\beta_{\mathrm{B}(\beta)}\neg\forall \mathrm{n}\neg \mathrm{A}(\alpha,\mathrm{n},\beta)$ so $\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{n}\mathrm{A}(\alpha,\mathrm{n},\beta)$ by (1); but then (c) $\exists \mathrm{x}\alpha(\mathrm{x}) = 0$.

4.2. **The Gödel-Dyson-Kreisel principle and weak completeness of HPC.** Verena Huber-Dyson and Kreisel verified in [13] that the *double negation* of (1) entails the *weak* completeness of **HPC** (in the sense that formulas which are not provable are not valid) for Beth's semantics, and Kreisel [37] proved the converse. The name "Gödel-Dyson-Kreisel Principle" seems appropriate for the schema

$$\forall\beta_{\mathrm{B}(\beta)}\neg\forall \mathrm{n}\neg \mathrm{A}(\mathrm{n},\beta) \to \neg\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{n}\mathrm{A}(\mathrm{n},\beta)$$

where $\mathrm{A}(\mathrm{n},\beta)$ expresses a primitive recursive relation of n,β. Primitive recursive relations depend only on finite initial segments of their choice sequence variables, so the principle can be expressed by the schema

$$\mathrm{GDK_{PR}}: \quad \forall\beta_{\mathrm{B}(\beta)}\neg\neg\exists \mathrm{n}\ \mathrm{u}(\overline{\beta}(\mathrm{n})) = 0 \to \neg\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{n}\ \mathrm{u}(\overline{\beta}(\mathrm{n})) = 0$$

where the u may represent any primitive recursive function.

From the point of view on intuitionistic analysis taken by Kleene and Vesley in [30] there seems to be no advantage in requiring the function represented by u to be primitive recursive, so consider also the principle

$$\mathrm{GDK}: \quad \forall\beta_{\mathrm{B}(\beta)}\neg\neg\exists \mathrm{n}\,\rho(\overline{\beta}(\mathrm{n})) = 0 \to \neg\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{n}\,\rho(\overline{\beta}(\mathrm{n})) = 0.$$

Theorem 1. *Assuming the double negation* $\neg\neg\mathrm{FT_1}$ *of the following form of the classically and intuitionistically correct binary fan theorem*[11]

$$\mathrm{FT_1}: \quad \forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x}\,\rho(\overline{\beta}(\mathrm{x})) = 0 \to \exists \mathrm{n}\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) = 0,$$

the double negation $\neg\neg\mathrm{WKL}$ *of weak König's Lemma*[12]

$$\mathrm{WKL}: \quad \forall \mathrm{n}\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) \neq 0 \to \exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0$$

is equivalent to GDK *over two-sorted intuitionistic arithmetic.*

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[11]While $\mathrm{FT_1}$ is false for recursive sequences β, it is true under the intended intuitionistic interpretation when β ranges over all binary choice sequences.

[12]By "the double negation of the fan theorem" we mean $\forall\rho\neg\neg\mathrm{FT_1}(\rho)$ rather than $\neg\neg\forall\rho\mathrm{FT_1}(\rho)$, and similarly for $\neg\neg\mathrm{WKL}$.

Proof. We can actually prove $\neg\neg \mathrm{FT}_1(\rho)$ & $\mathrm{GDK}(\rho) \leftrightarrow \neg\neg\mathrm{WKL}(\rho)$ in two-sorted intuitionistic arithmetic $\mathbf{IA}_1$, a weak subsystem of primitive recursive arithmetic obtained from Kleene's $\mathbf{I}$ by omitting the axiom schemas of countable choice, bar induction and continuous choice.[13] $\mathbf{IA}_1$ only proves the existence of primitive recursive functions so is weaker than **EL**.

First consider $\neg\neg\mathrm{FT}_1(\rho)$ & $\mathrm{GDK}(\rho) \rightarrow \neg\neg\mathrm{WKL}(\rho)$. Since $\neg\neg\mathrm{WKL}(\rho)$ is equivalent in $\mathbf{IA}_1$ to $\neg\neg\forall \mathrm{n}\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) \neq 0 \rightarrow \neg\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0$ it will be enough to prove $\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0 \rightarrow \neg\forall \mathrm{n}\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) \neq 0$.

Assume (a) $\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\, \rho(\overline{\beta}(\mathrm{x})) \neq 0$. Then (b) $\forall\beta_{\mathrm{B}(\beta)}\neg\neg\exists \mathrm{x}\, \rho(\overline{\beta}(\mathrm{x})) = 0$, so by $\mathrm{GDK}(\rho)$: (c) $\neg\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x}\, \rho(\overline{\beta}(\mathrm{x})) = 0$. Then (d) $\neg\neg\exists \mathrm{n}\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) = 0$ by $\neg\neg\mathrm{FT}_1(\rho)$, so (e) $\neg\forall \mathrm{n}\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) = 0$. Because there are only finitely many binary sequences of length $\leq n$, $\mathbf{IA}_1$ proves that $\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) = 0$ is equivalent to $\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) \neq 0$, which together with (e) gives the desired conclusion.

For the converse, assume (f) $\forall\beta_{\mathrm{B}(\beta)}\neg\neg\exists \mathrm{x}\, \rho(\overline{\beta}(\mathrm{x})) = 0$, so (equivalently in $\mathbf{IA}_1$) (g) $\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\, \rho(\overline{\beta}(\mathrm{x})) \neq 0$. Then (h) $\neg\forall \mathrm{n}\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\rho(\overline{\beta}(\mathrm{x})) \neq 0$ by $\neg\neg\mathrm{WKL}(\rho)$, and this is equivalent in $\mathbf{IA}_1$ to (i) $\neg\forall \mathrm{n}\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) = 0$ and hence to (j) $\neg\neg\exists \mathrm{n}\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) = 0$. Then *a fortiori* (k) $\neg\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x}\, \rho(\overline{\beta}(\mathrm{x})) = 0$, completing the proof of $\neg\neg\mathrm{WKL}(\rho) \rightarrow \mathrm{GDK}(\rho)$ in $\mathbf{IA}_1$. The proof in $\mathbf{IA}_1$ of $\neg\neg\mathrm{WKL}(\rho) \rightarrow \neg\neg\mathrm{FT}_1(\rho)$ is similar. □

This equivalence is implicit in [13]. As a consequence, **HPC** is weakly complete for Beth semantics if and only if weak König's Lemma is *persistently consistent* with the fan theorem in the sense that any counterexample to either would generate a counterexample to the other. Note that Kleene's counterexample for recursive sequences (Lemma 9.8 of [30]) applies equally to both.

5. Consistency and independence results

As it follows from MP_1, GDK is evidently consistent with $\mathbf{I}$.[14] Before showing that $\mathbf{I}$ + GDK lies strictly between $\mathbf{I}$ and $\mathbf{I}$ + MP, we consider Vesley's Schema:

$$\begin{aligned}\mathrm{VS}: \quad &\forall\alpha\forall \mathrm{x}\exists\beta(\overline{\beta}(\mathrm{x}) = \overline{\alpha}(\mathrm{x})\ \&\ \neg \mathrm{A}(\beta))\ \&\ \forall\alpha(\neg \mathrm{A}(\alpha) \rightarrow \exists\beta \mathrm{B}(\alpha,\beta)) \\ &\rightarrow \forall\alpha\exists\beta(\neg \mathrm{A}(\alpha) \rightarrow \mathrm{B}(\alpha,\beta)).\end{aligned}$$

In [69] Vesley proposed VS as a "palatable substitute" for Kripke's Schema KS, which proves Brouwer's "creating subject" counterexamples but conflicts with the $\forall\alpha\exists\beta$-continuity axiom schema of $\mathbf{I}$.[15] Vesley proved that $\mathbf{I}$ + VS is consistent (by special-realizability) and refutes a large number of classical principles objected to by Brouwer, including Markov's Principle.

Because it is special-realizable, GDK is consistent with $\mathbf{I}$ + VS. Assuming **HPC** is weakly complete for Beth semantics, GDK is a palatable substitute for Markov's Principle MP_1 and the formal system $\mathbf{I}$ + VS + GDK has real advantages over $\mathbf{I}$ as an axiomatization of intuitionistic mathematical practice.

[13] See [54] for a precise definition.

[14] MP_1 is interderivable with $\forall\rho[\forall\beta_{\mathrm{B}(\beta)}\neg\forall \mathrm{n}\neg\, \rho(\overline{\beta}(\mathrm{n})) = 0 \rightarrow \forall\alpha_{\mathrm{B}(\beta)}\exists \mathrm{n}\, \rho(\overline{\beta}(\mathrm{n})) = 0]$ over $\mathbf{IA}_1$, so $\mathbf{IA_1} + \mathrm{MP}_1 \vdash$ GDK. WKL is equivalent over $\mathbf{IA}_1$ to the conjunction of FT_1 and MP_1.

[15] KS is consistent with $\forall\alpha\exists$x-continuity and with $\forall\alpha\exists!\beta$-continuity, but KS and MP_1 together entail the law of excluded middle; cf. [56], [50].

Theorem 2. GDK *is consistent with* **I** *and with Vesley's Schema* VS*, which proves Brouwer's "creating subject" counterexamples. Hence* GDK *does not entail* MP_1*.*

Proof. GDK is both Kleene function-realizable and ${}_S$realizable. Vesley's Schema is ${}_S$realizable so **I** + VS + GDK is consistent. MP_1 is realizable but not ${}_S$realizable, so is not provable in **I** + VS + GDK. (In fact, **I** + VS + MP_1 is inconsistent.) □

Because it is not ^{G}realizable in the sense of the author's [49], GDK does not follow from weak Church's Thesis.

Theorem 3. GDK *is independent of* $\mathbf{I} + \mathrm{VS}$.

Proof. **I** + VS is consistent with "weak Church's thesis" $\forall\alpha\neg\neg\mathrm{GR}(\alpha)$, where $\mathrm{GR}(\alpha)$ is $\exists \mathrm{e}\forall \mathrm{x}\exists \mathrm{y}(\mathrm{T}(\mathrm{e},\mathrm{x},\mathrm{y})\ \&\ \mathrm{U}(\mathrm{y}) = \alpha(\mathrm{x}))$. The proof in [49] is by ^{G}realizability, and GDK is not ^{G}realizable. □

6. Markov's principle and order in the continuum

6.1. **MP and the intuitionistic continuum.** Brouwer's theory of the continuum is closely related to his treatment of Baire and Cantor space, and MP_1 can be expressed in terms of real numbers. Relying on sources by Brouwer and Heyting, in Chapter 3 of [30] Vesley gave a precise formal development of the nonnegative intuitionistic reals as equivalence classes, under coincidence, of *real number generators.* An r.n.g. is a Cauchy sequence $\alpha(0), \alpha(1)/2^1, \alpha(2)/2^2, \alpha(3)/2^3, \ldots$ of dual fractions, determined completely by the sequence α of numerators satisfying the Cauchy condition $\forall k\exists x\forall p 2^k|2^p\alpha(x) - \alpha(x+p)| < 2^{x+p}$.

A *canonical real number generator* (c.r.n.g.) is an r.n.g. satisfying the uniform Cauchy condition $\forall x|2\alpha(x) - \alpha(x+1)| \le 1$, which is much easier to work with formally. The fact that every r.n.g. coincides with a c.r.n.g. is provable using countable choice, but Vesley formalized a significant part of intuitionistic real analysis in terms of r.n.g. We follow his treatment, based on Brouwer [7], for the definitions and a few basic properties of four intuitionistic relations between real number generators: coincidence $\overset{\circ}{=}$, apartness $\#$, virtual ordering $\dot{<}$ and the measurable natural ordering $<\!\circ$.[16]

Each of the following formulas defines the relation on the left side of the $\leftrightarrow$.

- $\alpha \in \mathbb{R} \leftrightarrow \forall \mathrm{k}\exists \mathrm{x}\forall \mathrm{p} 2^{\mathrm{k}}|2^{\mathrm{p}}\alpha(\mathrm{x}) - \alpha(\mathrm{x}+\mathrm{p})| < 2^{\mathrm{x}+\mathrm{p}}$
- $\alpha \overset{\circ}{=} \beta \leftrightarrow \forall \mathrm{k}\exists \mathrm{x}\forall \mathrm{p} 2^{\mathrm{k}}|\alpha(\mathrm{x}+\mathrm{p}) - \beta(\mathrm{x}+\mathrm{p})| < 2^{\mathrm{x}+\mathrm{p}}$.
- $\alpha \,\#\, \beta \leftrightarrow \exists \mathrm{k}\exists \mathrm{x}\forall \mathrm{p} 2^{\mathrm{k}}|\alpha(\mathrm{x}+\mathrm{p}) - \beta(\mathrm{x}+\mathrm{p})| > 2^{\mathrm{x}+\mathrm{p}}$.
- $\alpha <\!\circ\, \beta \leftrightarrow \exists \mathrm{k}\exists \mathrm{x}\forall \mathrm{p} 2^{\mathrm{k}}(\beta(\mathrm{x}+\mathrm{p}) \dot{-} \alpha(\mathrm{x}+\mathrm{p})) \ge 2^{\mathrm{x}+\mathrm{p}}$.
- $\alpha \,\dot{<}\, \beta \leftrightarrow \neg\beta <\!\circ\, \alpha\ \&\ \neg\alpha \overset{\circ}{=} \beta$.
- $\alpha \in \mathbb{R}' \leftrightarrow \forall \mathrm{x}|2\alpha(\mathrm{x}) - \alpha(\mathrm{x}+1)| \le 1$.

Vesley derived hundreds of equivalences in $\mathbf{IA}_1$ plus countable choice, including

(1) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \,\dot{<}\, \beta \leftrightarrow \neg\neg\alpha <\!\circ\, \beta)$.

(2) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \overset{\circ}{=} \beta \leftrightarrow \neg\alpha \,\dot{<}\, \beta\ \&\ \neg\alpha \,\dot{>}\, \beta)$.

[16] Heyting (cf. [18], page 107) called the measurable natural ordering the "pseudo-ordering" and symbolized it by "$<$" instead of Brouwer's "$<\!\circ$." Bishop retained Heyting's symbol for the measurable natural order, which agrees with its classical use, but Mandelkern [45] introduced the term "pseudo-positive" for a (constructively) weaker relation he symbolized by "$0 <\!\circ$," and changed Heyting's "$\dot{<}$" for the virtual ordering to "$<\cdot$." The resulting notational disconnect with the careful treatment in [30] has probably resulted in much duplication of Vesley's and Kleene's work by modern constructivists.

(3) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \mathrel{\dot{<}} \beta \leftrightarrow \neg\alpha \mathrel{\dot{>}} \beta \;\&\; \neg\alpha \overset{\circ}{=} \beta)$.
(4) $\forall\alpha_{\alpha\in\mathbb{R}}\exists\beta_{\beta\in\mathbb{R}'}(\alpha \overset{\circ}{=} \beta)$.

Kleene ([30] Chapter 4) analyzed the logical relationships among various order properties of real numbers considered by Brouwer in [8].[17] He proved in particular that MP_1 is interderivable over $\mathbf{IA}_1$ with each of the following statements:[18]

(a) $\forall z\forall\alpha[\neg\forall y\neg \mathrm{T}^1_0(\widetilde{\alpha}\,(y), z, y) \to \exists y\mathrm{T}^1_0(\widetilde{\alpha}\,(y), z, y)]$
(b) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\neg\alpha \overset{\circ}{=} \beta \leftrightarrow \alpha \,\#\, \beta)$
(c) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \mathrel{\dot{<}} \beta \leftrightarrow \alpha <\!\circ\, \beta)$

Kleene also showed that $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\neg\alpha \overset{\circ}{=} \beta \to \alpha \,\#\, \beta)$ is equivalent over $\mathbf{IA}_1$ to $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \mathrel{\dot{<}} \beta \to \alpha <\!\circ\, \beta)$, and that neither is derivable in $\mathbf{I}$.

6.2. **MP and the constructive continuum.** Working informally and avoiding the use of negation, Bishop [5] represented constructive real numbers by *regular* Cauchy sequences $\{x_n\}$ of rational numbers satisfying $|x_n - x_m| < 1/n + 1/m$. He did not accept Markov's principle so the distinctions described above are meaningful for his constructive reals. Let $\mathbb{R}''$ be the class of regular Cauchy sequences $x = \{x_n\}$. It is straightforward to prove (using countable choice, which Bishop accepted) that every Cauchy sequence coincides with a regular one, so $\mathbb{R}$, $\mathbb{R}'$ and $\mathbb{R}''$ represent the same constructive reals.[19]

Mandelkern [45] showed that Markov's principle is equivalent to the conjunction of the "lesser limited principle of existence" LLPE and the "weak limited principle of existence" WLPE. He stated these principles using "$<$" instead of "$<\!\circ$" and "$<\cdot$" instead of "$\dot{<}$" but in Vesley's notation LLPE is $\forall\beta_{\beta\in\mathbb{R}''}(\neg\neg 0 <\!\circ\, \beta \to 0 \mathrel{\dot{<}} \beta)$; WLPE is $\forall\beta_{\beta\in\mathbb{R}''}\forall\alpha_{\alpha\in\mathbb{R}}(0 \mathrel{\dot{<}} \alpha \vee \alpha \mathrel{\dot{<}} \beta) \to 0 <\!\circ\, \beta)$; and the "limited principle of existence" LPE is $\forall\alpha_{\alpha\in\mathbb{R}}(0 \mathrel{\dot{<}} x \to 0 <\!\circ\, x)$, which is equivalent to MP_1 by [30].

In [46] Mandelkern derived equivalences between order properties of constructive real numbers and properties of the *decision sequences* (monotone nondecreasing binary sequences) involved in Brouwer's creating subject counterexamples. Then in [22] Ishihara decomposed MP_1 into "weak Markov's principle"

$$\mathrm{WMP}: \quad \forall\alpha[\forall\beta(\neg\neg\exists n(\beta(n) \neq 0) \vee \neg\neg\exists n(\alpha(n) \neq 0 \;\&\; \beta(n) = 0)) \to \exists n(\alpha(n) \neq 0)]$$

and "disjunctive Markov's principle"

$$\mathrm{MP}^{\vee}: \quad \forall\alpha\forall\beta[\neg\neg\exists n(\alpha(n) = 0 \vee \beta(n) = 0) \to \neg\neg\exists n\alpha(n) = 0 \vee \neg\neg\exists n\beta(n) \neq 0]$$

equivalent respectively to Mandelkern's WLPE and LLPE. Their main interest was to discover which theorems of classical real analysis could (only) be proved in their original form by assuming these and other principles considered to be nonconstructive. Much more has been done along these lines (cf. [6] and [23]).

7. The descriptive power of MP and MR in the context of analysis

Many studies of the consequences and relative independence of variants and weakenings of Markov's principle in the context of systems with intuitionistic logic have been published since Kreisel's [35]. Among these, in rough chronological order, are Myhill [56]; Vesley [70]; Troelstra [63]; Luckhardt [43] and [44]; Beeson

[17] The diagram on page 177 gives five equivalents of $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\neg\alpha \overset{\circ}{=} \beta \to \alpha \,\#\, \beta)$ over $\mathbf{I}$.

[18] (a) extends M_0 from recursive functions to recursive functionals, while (b) and (c) confirm that acceptance of MP_1 would greatly simplify the intuitionistic theory of the continuum.

[19] Vesley's treatment can easily be extended informally to all intuitionistic real numbers, so 0 loses its special status as a boundary point.

[3]; Mandelkern [45] and [46]; Bridges and Richman [6]; Ishihara [22] and especially [23]; Scedrov and Vesley [58]; Coquand and Hoffman [10]; Ishihara and Mines [24]; Kohlenbach [31] and [32]; Akama, Berardi, Hayashi and Kohlenbach [1]; Moschovakis [52]; Loeb [42]; Herbelin [17]; Ilik and Nakata [21]; Fujiwara, Ishihara and Nemoto [15]; Hendtlass and Lubarsky [16]; Coquand and Mannaa [11].

Even when obtained by classical methods, technical results of this kind can be interpreted in terms of the models permitted or eliminated by versions of MP, as already observed for $\mathrm{MP}_{\mathrm{QF}}$ in the context of arithmetic and GDK in the context of predicate logic. This section considers some consequences of adding MP_1, or a weak version of MP_1 such as $\mathrm{MP}^{\vee}$ or WMP or GDK, to two-sorted intuitionistic systems between $\mathbf{IA}_1$ and $\mathbf{I}$ which already obey Markov's rule.

7.1. **Strong existence and the Church-Kleene Rule.** $\forall\alpha\mathrm{GR}(\alpha)$ is obviously inconsistent with $\mathbf{I}$ by $\forall\alpha\exists x$-continuous choice. By Lemma 9.8 of [30] $\forall\alpha\mathrm{GR}(\alpha)$ is inconsistent with the classically correct subsystem $\mathbf{F} = \mathbf{IA}_1 + \mathrm{FT}_1$ of $\mathbf{I}$, so even without continuity principles Brouwer's choice sequences cannot all be recursive.

Nevertheless, as Kleene proved in [29], if $\exists\alpha\mathrm{A}(\alpha)$ is closed and $\mathbf{I} \vdash \exists\alpha\mathrm{A}(\alpha)$ then $\mathbf{I} \vdash \exists\alpha[\mathrm{GR}(\alpha)\ \&\ \mathrm{A}(\alpha)]$.[20] This *Church-Kleene Rule* holds also for $\mathbf{I} + \mathrm{MP}_1$; so $\mathbf{I}$ and $\mathbf{I} + \mathrm{MP}_1$ prove *strong* existence only for recursive sequences.

7.2. **Weak existence of arithmetical sequences.** The situation is different for *weak* existence. On the one hand, $\forall\alpha\neg\neg\mathrm{GR}(\alpha)$ is consistent with $\mathbf{I}$ by [49]. By ^{G}realizability it is also consistent with $\mathbf{I}$ extended by the principle

$$\forall\alpha[\forall \mathrm{x}\neg\neg\exists \mathrm{y}\alpha(\langle \mathrm{x},\mathrm{y}\rangle) = 0 \rightarrow \neg\neg\forall \mathrm{x}\exists \mathrm{y}\alpha(\langle \mathrm{x},\mathrm{y}\rangle) = 0],$$

a version of the consequence of MP studied in [58].

On the other hand, Solovay showed that $\mathbf{S} + \mathrm{MP}_1$ proves the weak existence of characteristic functions for all arithmetical relations (and therefore $\mathbf{S} + \mathrm{MP}_1 \vdash \neg\neg\exists\alpha\neg\mathrm{GR}(\alpha)$), where $\mathbf{S}$ is a classically correct subsystem of $\mathbf{I}$ including the bar induction schema BI_1 and countable choice for arithmetical relations.[21] A careful analysis of his argument shows that $\neg\neg\exists\alpha\neg\mathrm{GR}(\alpha)$ is also provable in $\mathbf{S}$ extended by the principle

$$\mathrm{DNS}_1: \quad \forall\rho[\forall\alpha\neg\neg\exists \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \neg\neg\forall\alpha\exists \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0]$$

which entails GDK but not MP_1. In $\mathbf{S} + \mathrm{DNS}_1$ (*a fortiori* in $\mathbf{I} + \mathrm{MP}_1$) we can prove that not all choice sequences are recursive even from the classical point of view. It is tempting to propose DNS_1 as an "as yet undiscovered" axiom which holds for the intended interpretation and entails GDK.

7.3. **WMP, $\mathbf{MP}^{\vee}$ and GDK.** WMP is provable in any subsystem of $\mathbf{I}$ extending $\mathbf{IA}_1$ and including $\forall\alpha\exists x$-continuous choice (cf. [23]), so MP_1 is equivalent to $\mathrm{MP}^{\vee}$ over $\mathbf{I}$. By Theorem 1 above, GDK is equivalent to $\neg\neg$WKL over $\mathbf{F}$, and this with [53] leads to another observation.

Theorem 4. WKL!! *(one form of Weak König's Lemma with uniqueness) is* WKL *with the additional hypothesis* $\forall\alpha\forall\beta(\forall \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) \neq 0\ \&\ \forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0 \rightarrow \alpha = \beta)$.

(1) WKL!! *is interderivable with* $\mathrm{MP}^{\vee} + \mathrm{GDK}$ *over* $\mathbf{F}$.
(2) WKL!! *is interderivable with* MP_1 *over* $\mathbf{I}$.

[20] In fact, under the same assumption there is a numeral $\mathbf{e}$ such that $\mathbf{I}$ proves a sentence which could be abbreviated by $\forall \mathrm{x}\{\mathbf{e}\}(\mathrm{x})\downarrow\ \&\ \mathrm{A}(\{\mathbf{e}\})$, cf. page 101 of [29].

[21] For his proof see [51].

7.4. **Work in progress.** The connection between weak existence and classical existence can be made precise by the Gödel-Gentzen negative interpretation. MP_1 and related principles, if added consistently to a formal system **T** of intuitionistic analysis, can express limitations on the ω-models of the neutral (classically correct) subsystem of **T**.

Together with the fact that the intuitionistic (and constructive) integers are the standard integers, this justifies a serious study of consequences of MP_1. Kleene proved that the arithmetical sequences (as opposed to the recursive sequences) form a classical ω-model of **F**, but that not even the hyperarithmetical sequences suffice for bar induction. Consequences of MP_1 (such as DNS_1 and GDK) allow us to express some of these distinctions formally.

8. Epilogue

We began this essay with the conviction that Markov's Principle, Markov's Rule and the common notion of constructive proof are legitimate candidates for investigation in the spirit of Kreisel's ideal of informal rigour. Following Heyting's implicit advice, our preliminary investigation was limited to the context of formal systems for intuitionistic predicate logic, arithmetic and analysis. However, informal rigour does not apply only to arguments formalizable in particular formal systems (cf. [47]). Brouwer stressed that to the extent formalization is justified at all, it must be preceded by the appropriate (informal intuitionistic) mathematics. We should not forget that a literal translation of "metamathematics" is "after mathematics."

Intuitionistic (as opposed to classical) formal systems have nonderivable admissible rules, such as Church's Rule for **HA** and the Church-Kleene Rule for **I**. For corresponding closed instances of Markov's Rule and Markov's Principle, the first may hold for an intuitionistic system which does not prove the second. We could call a formal system with intuitionistic logic *Markovian* if it satisfies an appropriate version of Markov's Rule, and agree to use only arguments which can in principle be formalized in Markovian systems. Then e.g. we could confidently assert that every (provably) Δ^0_1 relation is recursive, without accepting MP_0 as an axiom (cf. [51]). In effect, we would be substituting a broader (but equally precise) notion of constructive proof for the unimaginative "proof formalizable using intuitionistic logic."

Alternatively, $\mathrm{MP}_{\mathrm{QF}}$ could be added to intuitionistic arithmetic as a way of asserting that we have only the standard natural numbers. MP_1 could be added to intuitionistic analysis or its neutral subsystem (without increasing the stock of provably recursive functions), making it possible to prove that the constructive arithmetical hierarchy does not collapse. GDK could be added to intuitionistic analysis as a way of asserting the adequacy of intuitionistic predicate logic for the intended interpretation, with possibly interesting mathematical consequences.

While reverse mathematics – whether classical or constructive – depends on fixing a basic axiomatic system, intuitionistic mathematics is meant to be extended as new insights are achieved. Kleene's acceptance of strong continuous choice, Kripke's and Myhill's and Vesley's work inspired by Brouwer's creating subject arguments, and Kreisel and Troelstra's work on lawless sequences and their projections support this view. Surely Brouwer would agree that the notion of constructive proof is dynamic, evolving in time.

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