Embedding the Classical in the Intuitionistic Continuum

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Outline:

1. Intuitionistic vs. classical logic
2. Intuitionistic vs. classical first-order arithmetic
3. Intuitionistic vs. classical Baire space ("the continuum"):
    - Brouwer's choice sequences (infinitely proceeding sequences)
    - Classical sequences (arbitrary, completely determined)
    - Brouwer's lawlike sequences (definable, determined)
4. Embedding the classical continuum **R** in the intuitionistic continuum **FIM** via a 3-sorted system **FIRM**. Using logic and language to separate the recursive, constructive, intuitionistic, and classical components of the continuum.
5. Kreisel's intensionally lawless sequences **LS**
6. An extensional alternative: relatively lawless sequences **RLS**
7. A definably well ordered subset $(\mathcal{R}, \prec_{\mathcal{R}})$ of the continuum
8. Consistency of **FIRM**, assuming $\mathcal{R}$ is countable

Intuitionistic logic differs from classical logic in three ways:

1. All four intuitionistic connectives $\neg$, &, $\vee$ and $\rightarrow$ are needed for intuitionistic propositional logic because none can be defined in terms of the others.
2. Intuitionistic propositional logic replaces the classical laws of excluded middle $A \vee \neg A$ and double negation $\neg\neg A \rightarrow A$ by the law of contradiction $\neg A \rightarrow (A \rightarrow B)$.
3. Both $\forall$ and $\exists$ are needed for intuitionistic predicate logic, as neither can be defined in terms of the other and negation. $\neg\forall x \neg A(x) \leftrightarrow \neg\neg\exists x A(x)$ and $\neg\exists x \neg A(x) \leftrightarrow \forall x \neg\neg A(x)$ are valid intuitionistically, but the double negations cannot be eliminated.

Remark. $\neg\neg\exists$ and $\forall\neg\neg$ express the classical existential and universal quantifiers, respectively, in intuitionistic logic.

Intuitionistic first-order arithmetic has the same mathematical axioms as classical first-order arithmetic: axioms for equality, 0, successor, addition and multiplication, and the axiom schema of mathematical induction. The only difference is the restriction to intuitionistic logic. Formally, intuitionistic arithmetic **IA** is a proper subsystem of classical (Peano) arithmetic **PA**. However, Gentzen's and Gödel's negative translations showed that

"*the system of intuitionistic arithmetic and number theory is only apparently narrower than the classical one, and in truth contains it, albeit with a somewhat deviant interpretation.*" (Gödel 1933)

Gentzen's negative translation associates to each formula E of the language of arithmetic a formula E^g without $\vee$ or $\exists$, so that

1. If E contains no $\vee$ or $\exists$ then E^g is E.
2. $\vdash_{\mathbf{PA}} E \leftrightarrow E^g$.
3. $\vdash_{\mathbf{PA}} E \Leftrightarrow \vdash_{\mathbf{IA}} E^g$.

How Intuitionistic Logic Affects Consistency:

Intuitionistic propositional logic proves $\neg\neg(A \vee \neg A)$, so *every consistent* formal system based on intuitionistic logic is *consistent with every sentence* (closed formula) of the form $A \vee \neg A$. Brouwer wrote: "*Consequently, the theorems which are usually considered* as proved *in mathematics, ought to be divided into those that are* true *and those that are* non-contradictory."

Intuitionistic predicate logic does *not* prove $\neg\neg\forall x(A(x) \vee \neg A(x))$.

If intuitionistic arithmetic **IA** is consistent, then every arithmetical sentence of the form $\forall x(A(x) \vee \neg A(x))$ is consistent with **IA** because **IA** is contained in **PA**.

But if $A(x)$ is $\exists z T(x, x, z)$, expressing "the computation of $\{x\}(x)$ converges," then $\neg\forall x(A(x) \vee \neg A(x))$ is also consistent with **IA** (and in fact *true* in Russian recursive mathematics).

Intuitionistic logic permits divergent mathematical views.

"The Continuum": The points of the linear continuum can be represented by infinite sequences of natural numbers. Instead of studying the linear continuum directly, we focus on Baire space (the collection ω^ω of all such sequences with the finite initial segment topology), called "*the continuum*" from now on.

Brouwer's intuitionistic continuum consists of infinitely proceeding sequences or "*choice sequences*" α of natural numbers, generated by more or less freely choosing one integer after another. At each stage, the chooser may or may not specify restrictions (consistent with those already made) on future choices.

Since the first *n* values of a choice sequence α may be the *only* information available at the *n*th stage of its construction,

$\neg\forall\alpha(\forall x(\alpha(x) = 0) \vee \neg\forall x(\alpha(x) = 0))$ is *intuitionistically true*.

The classical continuum consists of all possible infinite sequences of natural numbers, each considered to be completely determined.

$\forall\alpha(A(\alpha) \vee \neg A(\alpha))$ is *classically true* for *every* $A(\alpha)$.

So while intuitionistic and classical arithmetic are equivalent, the intuitionistic continuum seems incompatible with the classical continuum. Still, there are areas of agreement, including

- 2-sorted primitive recursive arithmetic,
- the axiom of countable choice, and
- induction up to a countable ordinal (bar induction).

Kleene and Vesley, in "Foundations of Intuitionistic Mathematics" (1965), formalized the common core **B** using intuitionistic logic.

- An intuitionistic theory **FIM** results from **B** by adjoining a strong axiom schema of continuous choice, ensuring that every function defined on the entire intuitionistic continuum is continuous in the finite initial segment topology.
- The classical theory **C** results from **B** by strengthening the logic to classical logic. So **C** conflicts with **FIM**, but Kleene proved **FIM** is consistent relative to **B**. *Is his result optimal?*

How complex and how large is the intuitionistic continuum?

Brouwer called "*lawlike*" any choice sequence all of whose values are determined in advance according to some fixed law.
He did not specify which laws are to be permitted, but each lawlike sequence must be definable in some sense.

Brouwer distinguished three infinite cardinalities:

1. *denumerably infinite*, as the natural numbers.
2. *denumerably unfinished*, when "each element can be individually realized, and ... for every denumerably infinite subset there exists an element not belonging to this subset." (footnote to "Intuitionism and Formalism" [1912])
3. *nondenumerable*, as the intuitionistic continuum.

The collection of all lawlike sequences is denumerably unfinished.
If $b_0, b_1, b_2, \ldots$ is a lawlike sequence of lawlike sequences, the sequence $b^*(n) = b_n(n) + 1$ is lawlike and differs from every b_n.

Lawlike and classical sequences are similar in two important ways:

- The collection of all lawlike sequences does not have a lawlike enumeration, and the collection of all classical sequences does not have a classical enumeration.
- Lawlike sequences, like classical sequences, are considered to be individually realized and completely determined.

They differ in two important ways:

- Not every choice sequence is lawlike, in fact “most” choice sequences are incomplete objects.
- Properties of classical sequences, but not lawlike sequences, are assumed to obey the law of excluded middle (the LEM).

Kleene proved that the LEM for *arithmetical formulas* is consistent with his intuitionistic system **FIM** if **C** is consistent.

Is the LEM for *formulas concerning numbers and lawlike sequences only* consistent with the intuitionistic theory of the continuum?

Embedding Theorem. *(a)* A unified axiomatic theory **FIRM** of the continuum extends Kleene's **FIM** in a language with a third sort of variable over lawlike sequences. **FIRM** proves that every lawlike sequence is extensionally equal to a choice sequence, but not conversely. The subsystem of **FIRM** without lawlike sequence variables is **FIM**. A subsystem **R** of **FIRM** without choice sequence variables is a notational variant of **C**. The logic of **R** is classical.

(b) Assuming a certain definably well ordered subset $\mathcal{R}$ of the continuum is countable, a classical realizability interpretation establishes the consistency of an extension **FIRM**($\prec$) of **FIRM** in which $\prec$ well orders the lawlike sequences.

(c) The subsystem **IR** of **R** with intuitionistic logic expresses Bishop's constructive mathematics **BCM**, and Russian recursive mathematics **RRM** $=$ **IR** $+$ ECT $+$ MP is consistent with the subsystem of **FIRM** with intuitionistic logic exclusively.

Why can't we assume all lawlike sequences are recursive?

Kleene's formal language had variables $x, y, z, \ldots$ over numbers and $\alpha, \beta, \gamma, \ldots$ over choice sequences, but no special variables $a, b, c, \ldots$ over lawlike sequences. All the lawlike sequences he needed were recursive.

Brouwer seems to have expressed no opinion on Church's Thesis, although it is likely that he was aware of it.

Primitive recursive sequences are lawlike, so recursive sequences are lawlike by the *comprehension axiom*:

$\mathrm{AC}^R_{00}!$ $\quad \forall x \exists ! y A(x, y) \to \exists b \forall x A(x, b(x))$

for $A(x, y)$ with only number and lawlike sequence variables, where

$$\exists ! y B(y) \equiv \exists y B(y) \;\&\; \forall x \forall y (B(x) \;\&\; B(y) \to x = y).$$

In **R** with classical logic, by $\mathrm{AC}^R_{00}!$ *all classical analytic functions* (with sequence quantifiers ranging over the lawlike part of the continuum) are lawlike. *So why aren't all sequences lawlike*?

In "*Lawless sequences of natural numbers*," *Comp. Math.* (1968), Kreisel described a system **LS** of axioms for numbers, lawlike sequences $b, c, \ldots$ and *intensionally "lawless"* sequences $\alpha, \beta, \ldots$ in which *"the simplest kind of restriction on restrictions is made, namely some finite initial segment of values is prescribed, and, beyond this, no restriction is to be made"*.

- Equality (= identity) of lawless sequences is decidable, and distinct lawless sequences are independent.
- Every neighborhood contains a lawless sequence.
- The axiom of open data holds: If $A(\alpha)$ where α is lawless, then $A(\beta)$ for all lawless β in some neighborhood of α.
- Lawless sequences satisfy strong effective continuous choice: If $\forall\alpha\exists b A(\alpha, b)$ then for some lawlike b, e: e codes a *total* continuous function, b codes a sequence of sequences $(b)_n$, and $\forall\alpha A(\alpha, (b)_{e(\alpha)})$. Arbitrary choice sequences do not appear explicitly, but e must be defined on them too.

Troelstra, in “Choice Sequences: A Chapter of Intuitionistic Mathematics”(1977) and in Chapter 12 of “Constructivism in Mathematics: An Introduction” (Troelstra and van Dalen, 1988), analyzed and corrected the axioms of **LS**. Troelstra formulated

The Extension Principle: Every function defined (and continuous) on all the lawless sequences has a continuous total extension.

This justified strong effective continuous choice. He noted that identity is the *only* lawlike operation under which the class of lawless sequences is closed. He suggested that lawlike sequence variables *may* be interpreted as ranging over “the classical universe of sequences.” And he provided a detailed proof of Kreisel’s

Theorem: Every formula E of **LS** without free lawless sequence variables can be translated into an equivalent formula $\tau(E)$ with only number and lawlike sequence variables, so “lawless sequences can be regarded as a figure of speech.”

Relatively Lawless Sequences: an Extensional Alternative

In 1987-1996 I developed a system **RLS** of axioms for numbers, lawlike sequences $a, b, \ldots, h$ and choice sequences $\alpha, \beta, \ldots$ extending Kleene's **B**. An arbitrary choice sequence α is *defined* to be "*R-lawless*" (lawless relative to the class R of lawlike sequences) if *every lawlike predictor correctly predicts* α *somewhere*:

$$RLS(\alpha) \equiv \forall b(Pred(b) \to \exists x \alpha \in \overline{\alpha}(x) * b(\overline{\alpha}(x))),$$

where $\overline{\alpha}(0) = \langle\,\rangle$, $\overline{\alpha}(x+1) = \langle \alpha(0), \ldots, \alpha(x) \rangle$; $Pred(b)$ says that b maps finite sequence codes to finite sequence codes; $\alpha \in u$ says that u codes an initial segment of α of length $lh(u)$; and $u * v$ codes the concatenation of sequences with codes u, v.

Extensional equality between arbitrary R-lawless sequences α, β is *not* assumed to be decidable. Two R-lawless sequences α and β are *independent* if and only if their *merge* $[\alpha, \beta]$ is R-lawless, where $[\alpha, \beta](2n) = \alpha(n)$ and $[\alpha, \beta](2n+1) = \beta(n)$. (cf. Fourman)

RLS has logical axioms and rules for all three sorts of quantifiers and an inductive definition of *term* and *functor*. R-terms and R-functors are those without choice sequence variables.

The new mathematical axioms of **RLS** include two *density axioms*:

RLS1. $\forall w(Seq(w) \rightarrow \exists\alpha(RLS(\alpha)\ \&\ \alpha \in w))$,

RLS2. $\forall\alpha[RLS(\alpha) \rightarrow \forall w(Seq(w) \rightarrow \exists\beta(RLS([\alpha,\beta])\ \&\ \beta \in w))]$,

where $Seq(w)$ expresses that w codes a finite sequence of numbers.

Definition. A formula is *restricted* if its choice sequence quantifiers all vary over independent R-lawless sequences, so $\forall\alpha(RLS([\alpha,\beta]) \rightarrow B(\alpha,\beta))$ and $\exists\alpha(RLS([\alpha,\beta])\ \&\ B(\alpha,\beta))$ are restricted if $B(\alpha,\beta)$ is restricted and has no choice sequence variables free other than α,β.

For $A(x,y)$ restricted, with no free occurrences of choice sequence variables, **RLS** has the *lawlike comprehension axiom*

$AC^R_{00}!$ $\forall x\exists! yA(x,y) \rightarrow \exists b\forall xA(x,b(x))$.

For the axioms of *open data*

RLS3. $\forall\alpha[RLS(\alpha) \rightarrow (A(\alpha) \rightarrow$
$\exists y \forall\beta(RLS(\beta) \rightarrow (\beta \in \overline{\alpha}(y) \rightarrow A(\beta))))]$

and *effective continuous choice for R-lawless sequences*

RLS4. $\forall\alpha[RLS(\alpha) \rightarrow \exists b A(\alpha, b)] \rightarrow$
$\exists e \exists b \forall\alpha[RLS(\alpha) \rightarrow e(\alpha)\downarrow \ \& \ A(\alpha, (b)_{e(\alpha)})]$

and the *restricted law of excluded middle*

RLEM. $\forall\alpha[RLS(\alpha) \rightarrow A(\alpha) \vee \neg A(\alpha)]$

the $A(\alpha)$ and $A(\alpha, b)$ must be restricted, with no choice sequence variables free but α. The LEM *for formulas with only number and lawlike sequence variables* follows from RLEM by RLS1.

RLS1, $AC^R_{00}!$ and RLS4 entail *lawlike countable choice*:

AC^R_{01}. $\forall x \exists b A(x, b) \rightarrow \exists b \forall x A(x, (b)_x)$

for $A(x, b)$ restricted, with no choice sequence variables free.

AC^R_{01} entails $AC^R_{00}!$, and RLS3, AC^R_{01} and RLEM entail RLS4.

RLS proves:

- $\forall a \exists! \beta (\forall x\ a(x) = \beta(x))$. Every lawlike sequence is (extensionally) equal to an arbitrary choice sequence.
- $\forall \alpha [RLS(\alpha) \rightarrow \neg \exists b (\forall x\ b(x) = \alpha(x))]$. No *R*-lawless sequence is equal to a lawlike sequence.
- Independent *R*-lawless sequences are unequal.
- The *R*-lawless sequences are closed under prefixing an arbitrary finite sequence of natural numbers.
- If α is *R*-lawless and *b* is a lawlike injection with lawlike range, then $\alpha \circ b$ is *R*-lawless.
- The *R*-lawless sequences are dense in the continuum.
- Troelstra's extension principle fails. Every *R*-lawless sequence contains a (first) 1 but the constant 0 sequence doesn't.

FIRM (the common extension of **RLS** and **FIM**) proves that equality between arbitrary *R*-lawless sequences α, β is undecidable.

Theorem 1. Every restricted formula E with no arbitrary choice sequence variables free is equivalent in **RLS** to a formula $\varphi(E)$ with only number and lawlike sequence variables.

Proof: Like Troelstra's proof for **LS** but simpler. Kreisel and Troelstra needed a constant K_0 to represent the class of lawlike codes of continuous *total* functions. We can *define*

$$J_0(e) \equiv \forall u[Seq(u) \;\&\; \forall n < lh(u)(e(\overline{u}(n)) = 0) \rightarrow \\ \exists v(Seq(v) \;\&\; e(u * v) > 0)] \;\& \\ \forall u \forall v[Seq(u) \;\&\; Seq(v) \;\&\; lh(v) > 0 \;\&\; e(u) > 0 \rightarrow e(u * v) = 0].$$

Then **RLS** proves $\forall e(J_0(e) \leftrightarrow \forall \alpha(RLS(\alpha) \rightarrow e(\alpha)\downarrow))$, and the conclusion of effective continuous choice for R-lawless sequences is simplified to $\exists e \exists b[J_0(e) \;\&\; \forall \alpha(RLS(\alpha) \rightarrow A(\alpha, (b)_{e(\alpha)}))]$.

Note: $J_0(e)$ only requires $e(\alpha)$ to be defined for R-lawless α. The R-lawless sequences alone do not satisfy the bar theorem, and Troelstra's extension principle does not hold.

Definition. **R** is the subsystem of **RLS** obtained by restricting the language to number and lawlike sequence variables, omitting RLS1-4, replacing $\mathrm{AC}^R_{00}!$ by AC^R_{01}, and replacing RLEM by LEM.

For $B(w)$ and $A(w)$ without lawlike sequence variables, Brouwer's *bar theorem* is expressed by an axiom of Kleene's **B**:

$$\text{BI!} \quad \forall\alpha\exists! x B(\overline{\alpha}(x)) \ \&\ \forall w(Seq(w)\ \&\ B(w) \rightarrow A(w))\ \&\ \forall w(Seq(w)\ \&\ \forall s A(w * \langle s\rangle) \rightarrow A(w)) \rightarrow A(\langle\,\rangle).$$

BI (like BI! but omitting the !) conflicts with **FIM**, but for $B(w)$, $A(w)$ without choice sequence variables, **R** proves

$$\mathrm{BI}^R. \quad \forall a\exists x B(\overline{a}(x))\ \&\ \forall w(Seq(w)\ \&\ B(w) \rightarrow A(w))\ \&\ \forall w(Seq(w)\ \&\ \forall s A(w * \langle s\rangle) \rightarrow A(w)) \rightarrow A(\langle\,\rangle),$$

so **R** is just a notational variant of **C** ($=$ **B** $+$ $(A \vee \neg A)$), since **B** has the countable axiom of choice AC_{01} as an axiom.

Definition. **FIRM** is the common extension of **RLS** and **FIM** in the 3-sorted language.

Definition. **RLS**($\prec$) is the system resulting from **RLS** by extending the language to include prime formulas $u \prec v$ where u, v are functors, and adding axioms W0-W5:

W0. $\alpha = \beta \ \& \ \alpha \prec \gamma \rightarrow \beta \prec \gamma$.
$\beta = \gamma \ \& \ \alpha \prec \beta \rightarrow \alpha \prec \gamma$.

W1. $\forall a \forall b [a \prec b \rightarrow \neg b \prec a]$.

W2. $\forall a \forall b \forall c [a \prec b \ \& \ b \prec c \rightarrow a \prec c]$.

W3. $\forall a \forall b [a \prec b \vee a = b \vee b \prec a]$.

W4. $\forall a [\forall b (b \prec a \rightarrow A(b)) \rightarrow A(a)] \rightarrow \forall a A(a)$,

where $A(a)$ is any restricted formula, with no choice sequence variables free, in which b is free for a.

W5. $\alpha \prec \beta \rightarrow \neg \forall a \forall b \neg (\alpha = a \ \& \ \beta = b)$.

Definition. **FIRM**($\prec$) is the common extension of **RLS**($\prec$) and **FIM**, so **FIRM**($\prec$) is related to **FIRM** as **RLS**($\prec$) is to **RLS**.

Theorem 2. Iterating definability, quantifying over numbers and lawlike and independent R-lawless sequences, eventually yields a *definably well ordered subset* $(\mathcal{R}, \prec_{\mathcal{R}})$ of the classical continuum. *Assuming* $\mathcal{R}$ *is countable "from the outside,"*

(a) There is a classical model $\mathcal{M}(\prec_R)$ of **RLS**$(\prec)$, with $\mathcal{R}$ as the class of lawlike sequences.

(b) The class $\mathcal{R}LS$ of $\mathcal{R}$-lawless sequences of the model is disjoint from $\mathcal{R}$ and is Baire comeager in ω^ω, with classical measure 0.

(c) A classical realizability interpretation establishes the consistency of **FIRM**$(\prec)$ and hence of **FIRM**.

We outline the inductive definition of $(\mathcal{R}, \prec_{\mathcal{R}})$ and define $\mathcal{M}(\prec_R)$. For details see "Iterated definability, lawless sequences, and Brouwer's continuum," `http://www.math.ucla.edu/∼joan/` and the reference list for that preprint.

Definition. If $F(a_0, \ldots, a_{k-1}) \equiv \forall x \exists ! y E(x, y, a_0, \ldots, a_{k-1})$ is a restricted formula where x, y are all the distinct number variables free in E, and the distinct lawlike sequence variables $a_0, \ldots, a_{k-1}$ are all the variables free in F in order of first free occurrence, and if $A \subset \omega^\omega$, $\prec_A$ wellorders A, $\varphi \in \omega^\omega$ and $\psi_0, \ldots, \psi_{k-1} \in A$, then *$E$ defines φ over A from $\psi_0, \ldots, \psi_{k-1}$* if and only if when lawlike sequence variables range over A and choice sequence variables over ω^ω, $\prec$ is interpreted by $\prec_A$, and $a_0, \ldots, a_{k-1}$ by $\psi_0, \ldots, \psi_{k-1}$:

(i) F is true, and

(ii) for all $x, y \in \omega$: $\varphi(x) = y$ if and only if $E(\mathbf{x}, \mathbf{y})$ is true

Definition. **Def($A, \prec_A$)** is the class of all $\varphi \in \omega^\omega$ which are defined over $(A, \prec_A)$ by some E from some $\psi_0, \ldots, \psi_{k-1}$ in A.

Observe that $A \subseteq \mathrm{Def}(A, \prec_A)$, since $a(x) = y$ defines every $\varphi \in A$ over A from itself. We have to extend $\prec_A$ to a well ordering $\prec_A^*$ of $\mathrm{Def}(A, \prec_A)$ so the process can be iterated.

The classical model $\mathcal{M}(\prec_{\mathcal{R}})$ of **RLS**$(\prec)$:

An R-*formula* has no arbitrary choice sequence variables free. Let $E_0(x, y), E_1(x, y), \ldots$ enumerate all restricted R-formulas in the language $\mathcal{L}(\prec)$ containing free no number variables but x, y, where $E_0(x, y) \equiv a(x) = y$. For each i, let $F_i \equiv \forall x \exists ! y E_i(x, y)$.

For $\varphi, \theta \in \mathrm{Def}(A, \prec_A)$, set $\varphi \prec_A^* \theta$ if and only if $\Delta_A(\varphi) < \Delta_A(\theta)$ where $\Delta_A(\varphi)$ is the smallest tuple $(i, \psi_0, \ldots, \psi_{k-1})$ in the lexicographic ordering $<$ of $\omega \cup \bigcup_{k>o}(\omega \times A^k)$ determined by $<$ on ω and $\prec_A$ on A such that E_i defines φ over A from $\psi_0, \ldots, \psi_{k-1}$.

If $\varphi \in A$ then $\Delta_A(\varphi) = (0, \varphi)$, so $\prec_A$ is an initial segment of $\prec_A^*$.

Define $R_0 = \phi$, $\prec_0 = \phi$, $R_{\zeta+1} = \mathrm{Def}(R_\zeta, \prec_\zeta)$, $\prec_{\zeta+1} = \prec_\zeta^*$, and at limit ordinals take unions.

By cardinality considerations there is a least ordinal η_0 such that $R_{\eta_0} = R_{\eta_0+1}$. Let $\mathcal{R} = R_{\eta_0}$ and $\prec_{\mathcal{R}} = \prec_{\eta_0}$. $\mathcal{M}(\prec_{\mathcal{R}})$ is the natural classical model in which lawlike sequence variables range over $\mathcal{R}$.

Key lemmas for the proof that if $\mathcal{R}$ is countable then $\mathcal{M}(\prec_{\mathcal{R}})$ is a classical model of **RLS**($\prec$) with $\mathcal{R}$ as the lawlike sequences:

Lemma 1. If $\mathcal{R}$ is countable then

(i) There is an $\mathcal{R}$-lawless sequence, and

(ii) If α is $\mathcal{R}$-lawless there is a sequence β such that $[\alpha, \beta]$ is $\mathcal{R}$-lawless.

Lemma 2. If α is $\mathcal{R}$-lawless, so are $\langle n \rangle * \alpha$ for every natural number n and $\alpha \circ g$ for every injection $g \in \mathcal{R}$ whose range can be enumerated by an element of $\mathcal{R}$.

Lemma 3. If $A(\alpha)$ satisfies the axiom RLS3 of open data in $\mathcal{M}(\prec_{\mathcal{R}})$, so does $\neg A(\alpha)$.

We now appeal to the not uncommon set-theoretic assumption that every definably well ordered subset of ω^ω is countable.

$\mathcal{R}$ and $\prec_{\mathcal{R}}$ are definable over ω^ω with closure ordinal η_0. Let $\chi : \omega \times \omega \to \{0,1\}$ code a well ordering of type η_0 and let $\Gamma : \omega \to \mathcal{R}$ be a bijection witnessing simultaneously the countability of $\mathcal{R}$ and (via χ) the order of generation of its elements, so that for each $n, m \in \omega$:

$$\Gamma(n) \prec_{\mathcal{R}} \Gamma(m) \Leftrightarrow \chi(n,m) = 1.$$

A Γ-*interpretation* $\mathit{\Psi}$ of a list $\Psi = x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_k, a_1, \ldots, a_m$ of distinct variables is any choice of n numbers, k elements of ω^ω and m numbers $r_1, \ldots, r_m$. Then $\Gamma(\mathit{\Psi})$ is the corresponding list of n numbers, k sequences and m elements $\Gamma(r_1), \ldots, \Gamma(r_m)$ of $\mathcal{R}$.

Lemma 4. To each list Ψ of distinct *number and lawlike sequence* variables and each restricted R-formula $A(x,y)$ containing free at most Ψ, x, y where $x, y, a \notin \Psi$, there is a partial function $\xi_A(\mathit{\Psi})$ so that for each Γ-interpretation $\mathit{\Psi}$ of Ψ: If $\forall x \exists ! y A(x,y)$ is true-$\Gamma(\mathit{\Psi})$ then $\xi_A(\mathit{\Psi})$ is defined and $\forall x A(x, a(x))$ is true-$\Gamma(\mathit{\Psi}, \xi_A(\mathit{\Psi}))$.

The Γ*-realizability interpretation of* **FIRM**($\prec$):

For $\pi \in \omega^\omega$, E a formula of $\mathcal{L}(\prec)$ with at most the distinct variables Ψ free, and Ψ a Γ-interpretation of Ψ, define π Γ-**realizes**-Ψ E by induction on the logical form of E:

- π Γ-realizes-Ψ a prime formula P, if P is true-Γ(Ψ).
- π Γ-realizes-Ψ A & B, if $(\pi)_0$ Γ-realizes-Ψ A and $(\pi)_1$ Γ-realizes-Ψ B.
- π Γ-realizes-Ψ $A \vee B$, if $(\pi(0))_0 = 0$ and $(\pi)_1$ Γ-realizes-Ψ A, or $(\pi(0))_0 \neq 0$ and $(\pi)_1$ Γ-realizes-Ψ B.
- π Γ-realizes-Ψ $A \to B$, if, if σ Γ-realizes-Ψ A, then $\{\pi\}[\sigma]$ Γ-realizes-Ψ B.
- π Γ-realizes-Ψ $\neg A$, if π Γ-realizes-Ψ $A \to 1 = 0$.
- π Γ-realizes-Ψ $\forall x A(x)$, if $\{\pi\}[x]$ Γ-realizes-Ψ, x $A(x)$ for each $x \in \omega$.
- π Γ-realizes-Ψ $\exists x A(x)$, if $(\pi)_1$ Γ-realizes-$\Psi, (\pi(0))_0$ $A(x)$.

- π Γ-realizes-Ψ $\forall a A(a)$, if $\{\pi\}[r]$ Γ-realizes-Ψ, r $A(a)$ for each $r \in \omega$.
- π Γ-realizes-Ψ $\exists a A(a)$, if $(\pi)_1$ Γ-realizes-$\Psi, (\pi(0))_0$ $A(a)$.
- π Γ-realizes-Ψ $\forall \alpha A(\alpha)$, if $\{\pi\}[\alpha]$ Γ-realizes-Ψ, α $A(\alpha)$ for each $\alpha \in \omega^\omega$.
- π Γ-realizes-Ψ $\exists \alpha A(\alpha)$, if $(\pi)_1$ Γ-realizes-$\Psi, \{(\pi)_0\}$ $A(\alpha)$.

Theorem 2 (c) (restated). Assuming Γ enumerates $\mathcal{R}$ as above, every closed theorem of **FIRM**($\prec$) is Γ-realized by some function and hence **FIRM**($\prec$) is consistent. So the classical continuum can consistently be viewed as the lawlike part of Brouwer's continuum.

Theorem 3. If $\mathcal{R}$ is countable, the $\mathcal{R}$-lawless sequences are all the generic sequences with respect to properties of finite sequences of natural numbers which are definable with parameters over $(\omega, \mathcal{R})$ by formulas with only number and lawlike sequence variables.

In a nutshell:

- *R-lawless* and *random* are orthogonal concepts, since a random sequence of natural numbers should possess certain regularity properties (e.g. the percentage of even numbers in its nth initial segment should approach .50 as n increases) while an R-lawless sequence will possess none.
- Brouwer's continuum satisfies the bar theorem, countable choice and continuous choice.
- The lawlike sequences satisfy the bar theorem and countable choice, but not continuous choice.
- The R-lawless sequences satisfy open data and restricted continuous choice, but not the bar theorem.
- The recursive sequences satisfy recursive countable choice but not the bar theorem or even the fan theorem.
- Bishop's constructive sequences satisfy countable choice.