CLASSICAL AND CONSTRUCTIVE HIERARCHIES IN EXTENDED INTUITIONISTIC ANALYSIS

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Abstract. This paper introduces an extension $\mathcal{A}$ of Kleene's axiomatization of Brouwer's intuitionistic analysis, in which the classical arithmetical and analytical hierarchies are faithfully represented as hierarchies of the domains of continuity. A *domain of continuity* is a relation $R(\alpha)$ on Baire space with the property that every constructive partial functional defined on $\{\alpha : R(\alpha)\}$ is continuous there. The domains of continuity for $\mathcal{A}$ coincide with the *stable* relations (those equivalent in $\mathcal{A}$ to their double negations), while *every* relation $R(\alpha)$ is equivalent in $\mathcal{A}$ to $\exists\beta A(\alpha,\beta)$ for some stable $A(\alpha,\beta)$ (which belongs to the classical analytical hierarchy).

The logic of $\mathcal{A}$ is intuitionistic. The axioms of $\mathcal{A}$ include countable comprehension, bar induction, Troelstra's generalized continuous choice, primitive recursive Markov's Principle and a classical axiom of dependent choices proposed by Krauss. Constructive dependent choices, and constructive and classical countable choice, are theorems. $\mathcal{A}$ is maximal with respect to classical Kleene function realizability, which establishes its consistency. The usual disjunction and (recursive) existence properties ensure that $\mathcal{A}$ preserves the constructive sense of "or" and "there exists."

1. Introduction

Constructive mathematics, as proposed by Bishop in the spirit of Kronecker, is a proper part of classical mathematics. Intuitionistic mathematics includes a larger part of classical mathematics, together with Brouwer's principle of continuous choice. Philosophical considerations aside, neither approach has been popular with working mathematicians, even though constructive mathematics studies methods of effective approximation crucial to classical applied mathematics while much of classical analysis is concerned with functions uniformly continuous on compact sets.

The first barrier to understanding the constructive approach is its basis in intuitionistic logic, which rejects the classical law of excluded middle in favor of the principle that a contradiction implies everything. Gödel [5] defined a *double negation translation* which associates uniformly with every theorem of classical number theory a classically equivalent theorem of intuitionistic number theory; for example, "A or not A" translates to the intuitionistically provable "not

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both A and not A." Thus intuitionistic arithmetic contains a virtual image of, and is equiconsistent with, classical arithmetic.

Any attempt to extend this method to analysis runs up against the problem that the double negation translation of the axiom of choice is not constructively valid. Most constructivists do accept the axiom of choice for collections of inhabited (as opposed to non-empty) sets;[1] Bishop, following Brouwer, says in [3]: "A choice function exists in constructive mathematics, because a choice is *implied by the very meaning of existence*."[2] In contrast, the double negation translation of the axiom of choice for countably infinite collections is independent of the principles of constructive mathematics (including the countable axiom of choice itself). In intuitionistic analysis the situation is even worse: Brouwer's principle entails the axiom of choice for collections indexed by ${}^{\omega}\omega$, but contradicts the double negation translation of this axiom.

Kleene and Vesley's axiomatization and development [11] of intuitionistic analysis aimed to demystify Brouwer's ideas, using general recursive functions to interpret the principles of countable choice, constructive bar induction, and continuous choice on spreads. Although the intuitionistic theory **FIM** of [11] conflicts with classical logic, Kleene's function-realizability proved **FIM** consistent relative to a common subtheory **B** of intuitionistic and classical analysis.

The considerable effort of formalizing the theory of partial recursive functionals, function-realizability, and its variant q-realizability, Kleene carried out in [10] using a minimal subtheory (call it **M**) of **B**. In addition to tying down his relative consistency proof, this work established that **B**, **FIM**, and other suitable extensions of **M** obey a recursive uniformization rule: If $\forall\alpha\exists\beta R(\alpha, \beta)$ is a closed theorem of the extended theory **S**, then there is a recursive total functional $\Phi[\alpha]$ for which **S** proves $\forall\alpha R(\alpha, \Phi[\alpha])$. In [9] and [10] he proposed using these methods to identify theorems of *classical* analysis which may be added as axioms to **FIM**, to obtain stronger mixed theories **S** which are consistent relative to their classically correct subtheories and satisfy the rule. Troelstra [17] obtained a *non*classical extension **T** of **FIM** with the same properties.

The main point of the following pages is to propose an informal axiomatic theory $\mathcal{A}$, consistently extending constructive and intuitionistic analysis, containing a virtual image of classical analysis, and closed under Kleene's rule. Within $\mathcal{A}$ one can develop naturally the classical and constructive arithmetical and analytical hierarchies, with the advantage that Kleene's rule extends to provide recursive partial choice functionals on classical domains (*domains of continuity*) such as the collection of all neighborhood functions for total continuous functionals.

[1] A set is *inhabited* if it has an element; *non-empty* if the assumption that it has no element leads to a contradiction. With the constructive meaning of existence these notions are distinct.

[2] Diaconescu [4] proved that the law of excluded middle follows from the axiom of choice for subfinite collections of inhabited subsets of $\{0, 1\}$. Beeson [2] observed that there is no continuous (in the natural topology) choice function for the collection of unordered pairs of antipodal points on the unit circle. These examples explain why constructive choice is stated in the context of an indexed collection of inhabited sets, providing a choice function whose domain is the (constructively understood) index set rather than the collection itself. This is the form which gives rise naturally to the axiom of dependent choices, in the special case of an indexed collection of inhabited subsets of the index set. Cf. Aczel's presentation axiom for constructive set theory.

Using Brouwer's principle, Veldman has shown that the constructive hierarchy, which includes only positive relations, collapses at Σ^1_2. In $\mathcal{A}$ there are relations which belong neither to the classical nor to the constructive hierarchy, but *every* relation $R(\alpha)$ is equivalent to $\exists\beta A(\alpha,\beta)$ for some $A(\alpha,\beta)$ belonging to the classical hierarchy. This extended hierarchy does not collapse.

The only nonclassical axiom of $\mathcal{A}$, due to Troelstra, depends on a syntactically defined class of relations isolated by Kleene. A formula is *almost negative* if it contains no disjunction, and no existential quantifier except immediately before a prime formula. In [9] (with [11]) Kleene showed that if E is any formula in which the scope of every universal function quantifier, and that of every implication, is almost negative, then E is constructively equivalent to the assertion that some function realizes E.

Completing this analysis, Troelstra [17] proposed a *generalized continuity principle* GC_1 relativizing Brouwer's principle of continuous choice to an almost negative hypothesis. Troelstra's Characterization Theorem (for function realizability) says that in the resulting consistent extension $\mathbf{T} = \mathbf{B} + \mathrm{GC}_1$ of **FIM**, *every* formula E is equivalent to the assertion that some function realizes E; and an arbitrary formula E is provable in $\mathbf{T}$ just in case $\mathbf{B}$ proves that some function realizes E. By [15] the classical equivalent of $\mathbf{B}$ proves that E has a realizing function, if and only if the double negation of E is provable in the theory obtained from $\mathbf{T}$ by adding two principles from classical logic: Kuroda's Principle for Numbers DNS_0 and Markov's Principle MP_{D} (or $\mathrm{MP}_{\mathrm{PR}}$; see the next section). This theory $\mathbf{T}^+$ is consistent relative to its classically correct subtheory and (like $\mathbf{T}$ itself) satisfies Kleene's rule.

Peter Krauss [12] proposed using double negation to give a constructive interpretation of classical mathematics, including "classical" versions of axioms of choice and dependent choices.[3] Brouwer's principle conflicts with some forms, but the classical axiom $\mathrm{AC}^\bullet_1$ of countable choice is provable in $\mathbf{T} + \mathrm{DNS}_0$; in fact, $\mathbf{T}^+ = \mathbf{T} + \mathrm{MP}_{\mathrm{PR}} + \mathrm{AC}^\bullet_1$. Strengthening $\mathrm{AC}^\bullet_1$ consistently to classical dependent choices $\mathrm{DC}^\bullet_1$ preserves Kleene's rule and Troelstra's Characterization Theorem. The resulting extension $\mathbf{A}$ of $\mathbf{T}^+$ proves $\neg\neg E$, if and only if the classical version of $\mathbf{B} + \mathrm{DC}_1$ proves that a realizing function for E exists.

Thus we are led to the axiomatic theory $\mathcal{A} = \mathcal{B} + \mathrm{MP}_{\mathrm{PR}} + \mathrm{DC}^\bullet_1 + \mathrm{GC}_1$, treated here in the spirit of informal rigor.[4] The robustness of Kleene's method encourages further additions; for example, the classical axiom $\mathrm{PD}^\bullet$ of projective determinacy is consistent with $\mathcal{A}$ (and $\mathcal{A} + \mathrm{PD}^\bullet$ satisfies Kleene's rule), if the classical version of $\mathcal{B}$ is consistent with dependent choices and projective determinacy.[5]

2. Preliminaries

The setting is Baire space ${}^\omega\omega$ with the finite-initial-segment topology. We consider relations on finite Cartesian products of copies of ω and ${}^\omega\omega$, distinguishing between a relation and its double negation. Every proof presented

[3] Similarly, Troelstra [18] studied "classical" real numbers defined by double negation.

[4] Calligraphic capitals will consistently denote informal theories; thus $\mathcal{B}$ is the informal axiomatic theory corresponding to $\mathbf{B}$.

[5] Interestingly, a constructively stronger form PD of projective determinacy holds intuitionistically for games on ${}^\omega 2$, but not for games on ${}^\omega\omega$; see Veldman [20] for the proofs.

here can be formalized in an applied two-sorted intuitionistic predicate logic with decidable equality between terms of type 0, a sufficient finite list of basic primitive recursive function constants with their axioms, full induction over ω, Church's λ (not essential, but convenient) with the appropriate formation rules and reduction axioms, and the countable comprehension axiom

$$\mathrm{AC}_0! \qquad \forall x \exists! y A(x,y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)),$$

from which follows the apparently stronger form

$$\mathrm{AC}_1! \qquad \forall x \exists! \alpha A(x,\alpha) \rightarrow \exists \alpha \forall x A(x, \lambda t.\alpha((x,t))).$$

This is the minimal theory (1,0,0) of [13] with Kleene's list of function constants from [10] with [11]; call the corresponding informal theory $\mathcal{M}$.[6]

Remark 2.1. *All (logical or mathematical) axioms beyond those of $\mathcal{M}$ will be stated as additional hypotheses, when required for the proof of a theorem. To help the reader accustomed to classical logic, Lemmas 3.5, 3.6, 3.14, 3.20 and 5.1 are lists of equivalences, and implications whose converses are unprovable intuitionistically, which hold informally in $\mathcal{M}$.*

In the spirit of informal rigor, let $x, y, \ldots$ vary over elements of ω, and $\alpha, \beta, \ldots$ over *choice sequences* (elements of ${}^{\omega}\omega$). Finite sequences, and basic neighborhoods in Baire space, are coded using primitive recursive functions as in [10]: Standard surjections map every finite sequence $x_0, \ldots, x_{k-1}$ of numbers to a number $x = (x_0, \ldots, x_{k-1})$ such that $x \geq k$ and $x \geq x_i$ for $0 \leq i < k$, and every finite sequence $\alpha_0, \ldots, \alpha_{m-1}$ of choice sequences to a single choice sequence $\alpha = \lambda z.(\alpha_0(z), \ldots, \alpha_{m-1}(z))$. The inverse mappings $(x)_i$ satisfy $(x_0, \ldots, x_{k-1})_i = x_i$ if $i < k$, or 0 if $i \geq k$, and similarly for $(\alpha)_j$.

Every number w can be interpreted as coding $lh(w)$ values of a function α, where $lh(w)$ is the number of distinct $j < w$ such that $(w)_j > 0$; thus

$$w|\alpha \ \equiv\ \forall j < w[(w)_j > 0 \rightarrow (w)_j = \alpha(j) + 1].$$

In particular, $\overline{\alpha}(x) \equiv (\alpha(0)+1, \ldots, \alpha(x \dot{-} 1)+1)$ codes the first x values of α when $x > 0$, with $\overline{\alpha}(0) \equiv 1$ coding the empty sequence; so $\overline{\alpha}(x)|\beta$ if and only if $\overline{\beta}(x) = \overline{\alpha}(x)$. Call w a *sequence number*, abbreviated $Seq(w)$, if $(w)_j > 0$ for all $j < lh(w)$; then $Seq(\overline{\alpha}(x))$ holds for all α, x. A primitive recursive *concatenation function* $*$ satisfies $u * v = ((u)_0, \ldots, (u)_{lh(u) \dot{-} 1}, (v)_0, \ldots, (v)_{lh(v) \dot{-} 1})$ if $Seq(u)$ and $Seq(v)$.

In [10] Kleene associates with each recursive derivation of a partial functional $\varphi(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)$ a numerical index f coding k, m, and a computation tree algorithm, so that if $w_1|\alpha_1, \ldots, w_m|\alpha_m$ where $x_1, \ldots, x_k, w_1, \ldots, w_m$ suffice to compute the value of $\varphi(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)$ using the instructions coded by f, then $\{f\}(x_1, \ldots, x_k, w_1, \ldots, w_m)$ codes a computation of that value (which is $\{f\}(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)$), otherwise $\{f\}(x_1, \ldots, x_k, w_1, \ldots, w_m)$ codes an unsuccessful attempt at a computation. Kleene's S^m_n, recursion, and normal form theorems hold for this coding with constructive proofs.

Using this coding, every theorem E of intuitionistic analysis $\mathcal{FIM}$ can be *realized*, provably in a classically correct subtheory $\mathcal{B}^-$ ($\mathcal{M}$ plus the principle of bar induction) of $\mathcal{FIM}$, by a recursive functional; hence $\mathcal{FIM}$ is consistent

[6] The basic formal theory **B** of [11] strengthens countable comprehension to countable choice AC_1 and adds the axiom schema of bar induction.

relative to $\mathcal{B}^-$. Moreover, the same functional q-*realizes* E, provably in $\mathcal{FIM}$; so $\mathcal{FIM}$ has the following constructive closure property.[7]

Kleene's Rule. *If $A(\alpha, x, \beta)$ is a relation on ${}^\omega\omega \times \omega \times {}^\omega\omega$ with no additional choice sequence parameters, and if $\forall\alpha\forall x\exists\beta A(\alpha, x, \beta)$, then there is a standard general recursive functional $\Phi : {}^\omega\omega \times \omega \to {}^\omega\omega$ such that $\forall\alpha\forall x A(\alpha, x, \Phi[\alpha, x])$.*[8]

In fact, *every* extension of $\mathcal{M}$ by classically correct axioms or axiom schemas $\mathrm{A}_1, \dots, \mathrm{A}_j$ which prove their own realizability and q-realizability, and nonclassical axioms or schemas $\mathrm{C}_1, \dots, \mathrm{C}_k$ such that

(i) $\mathcal{M} + \mathrm{A}_1 + \dots + \mathrm{A}_j$ proves that each C_i is realizable, and
(ii) $\mathcal{M} + \mathrm{A}_1 + \dots + \mathrm{A}_j + \mathrm{C}_1 + \dots + \mathrm{C}_k$ proves that each C_i is q-realizable,

is consistent relative to $\mathcal{M} + \mathrm{A}_1 + \dots + \mathrm{A}_j$ and satisfies Kleene's Rule. Every theory in which we work here is of this type.

Any theory closed under Kleene's Rule has the property that every relation $A(\alpha, x)$ which is *decidable*, in the sense that $\forall\alpha\forall x[A(\alpha, x) \vee \neg A(\alpha, x)]$ is provable, is recursive according to the theory. This gives a constructive flavor to the theory even if it includes nonconstructive axioms.[9]

This brings up the matter of *Markov's Principle*, whose variations in the language of analysis include

$$\mathrm{MP_D} \qquad \forall\alpha[\forall x(A(\alpha, x) \vee \neg A(\alpha, x)) \wedge \neg\forall x \neg A(\alpha, x) \to \exists x A(\alpha, x)],$$

$$\mathrm{MP_{PR}} \qquad \forall y \forall\alpha[\neg\forall x \neg R(\overline{\alpha}(x), y) \to \exists x R(\overline{\alpha}(x), y)] \quad (\text{R primitive recursive}),$$

$$\mathrm{MP_1} \qquad \forall\alpha[\neg\forall x \neg\alpha(x) = 0 \to \exists x \alpha(x) = 0].$$

Most constructivists reject MP, although each version stated here entails its own realizability and q-realizability. All three are equivalent in $\mathcal{M}$ (cf. [11] pp. 185-6). Optimistically accepting the principle of unbounded search, our theory $\mathcal{A}$ includes $\mathrm{MP_{PR}}$ among its axioms; hence all three hold for $\mathcal{A}$.

Any theory which proves $\mathrm{MP_D}$ evidently satisfies

Markov's Rule MR. *If $\forall\alpha\forall x(A(\alpha, x) \vee \neg A(\alpha, x))$ and $\forall\alpha\neg\forall x \neg A(\alpha, x)$ hold, so does $\forall\alpha\exists x A(\alpha, x)$.*

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[7] For readability, most metamathematical details are omitted from this paper. Theorem 50, Corollary 57, and Sections 5.6-5.10 of [10] justify all our uses of realizability and q-realizability. See [15] for a summary and relevant applications of this method.

[8] By "ϕ (or Φ) is a standard general recursive function(al)" we mean that the proof that ϕ (or Φ) is total depends only on the classically correct axioms of the theory. Coding extends Kleene's Rule to $A(\alpha_1, \dots, \alpha_m, x_1, \dots, x_k, \beta)$ (and also to $A(\alpha_1, \dots, \alpha_m, x_1, \dots, x_k, x)$), in which case $\Phi : {}^\omega\omega \times \dots \times {}^\omega\omega \times \omega \times \dots \times \omega \to {}^\omega\omega$ (or $\varphi : {}^\omega\omega \times \dots \times {}^\omega\omega \times \omega \times \dots \times \omega \to \omega$). The rule relativizes to parameters $\beta_1, \dots, \beta_n$, with Φ (or φ) recursive in $\beta_1, \dots, \beta_n$.

[9] Kleene's Rule implies that every proof of a disjunction $A \vee B$ without parameters provides a guide to a proof of A or a proof of B; every proof of $\exists x A(x)$ without parameters can be improved to a proof of $A(n)$ for some particular number n; and every proof of $\exists\alpha A(\alpha)$ without parameters can be improved to a proof of $A(\{e\})$ for a particular Gödel number e of a standard provably recursive function. However, if MP (or any other classical axiom not accepted by intuitionists) was used in the original proof, then the choice between A and B, or the identification of an appropriate n or e, may not be intuitionistically effective.

$\mathcal{M}$, $\mathcal{B}^-$, $\mathcal{B}$, $\mathcal{FIM}$ and $\mathcal{T}$, none of which proves $\mathrm{MP_D}$, also satisfy Markov's Rule.[10] It seems to be an open question whether the rule holds for the subtheory of $\mathcal{A}$ without $\mathrm{MP_{PR}}$, but it certainly holds for $\mathcal{A}$.

Remark 2.2. *In what follows, both Kleene's and Markov's Rules will be used freely. Thus the results of Section 3, proved in $\mathcal{M}$ using both rules, hold also for $\mathcal{T}$ and $\mathcal{A}$, but perhaps not for the subtheory of $\mathcal{A}$ without* $\mathrm{MP_{PR}}$.

Kleene's and Markov's Rules together imply that if $A(\alpha, x, y)$ is decidable, and $\forall\alpha\forall x\neg\neg\exists y A(\alpha, x, y)$, then there is a φ recursive in the choice sequence parameters of A such that $\forall\alpha\forall x A(\alpha, x, \phi(\alpha, x))$. In fact, $\varphi(\alpha, x) = \mu y A(\alpha, x, y)$ is such a function.[11]

Equality between numbers is primitive and decidable, intuitionistically as well as classically. *By "equation" we always mean an equation between the values of primitive recursive number-valued functions, so all equations are (recursively) decidable.* Equality between choice sequences, defined extensionally by

$$\alpha = \beta \;\equiv\; \forall x(\alpha(x) = \beta(x)),$$

is undecidable in the intuitionistic theory (there is no continuous decision procedure for equality in Baire space). However, $\forall\alpha\forall\beta\neg\neg((\alpha = \beta) \vee \neg(\alpha = \beta))$ holds intuitionistically, and classically expresses the decidability of equality, so there is no real problem here. Rather, as Gödel [5] recognized in the context of first-order arithmetic, there is an opportunity.

3. The Arithmetical Hierarchies

First we define and compare the constructive and classical arithmetical hierarchies using only reasoning which is correct both constructively and classically, without bar induction or any choice principles (though the countable comprehension axiom $\mathrm{AC_0}!$, and Kleene's and Markov's Rules, are assumed).[12] The necessary properties of general recursive functions are available by [10].

Definition 3.1. *A relation $R(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)$ is* recursive *if and only if R has a general recursive characteristic function χ_R.*

Intuitionistically, only those relations which are continuous in their choice sequence variables can have total characteristic functions (which may or may

[10] The Friedman-Dragalin translation, in conjunction with Kleene's Rule, justifies MR in each case; see [19] or [1] for details. Translating by $\neg\neg Q$ proves that these theories, as well as the subtheory of $\mathcal{A}$ without the axiom $\mathrm{MP_{PR}}$, also satisfy the "independence of premise" rule $\mathrm{IPR_1}$: If $\neg A \rightarrow \exists\beta B(\beta)$ holds, where A does not depend on β, then $\exists\beta(\neg A \rightarrow B(\beta))$ holds. Since it conflicts with $\mathrm{IPR_1}$, $\mathrm{MP_{PR}}$ is independent of the other axioms of $\mathcal{A}$. The behavior of relations $\neg A(\alpha)$ as domains of continuity for $\mathcal{A}$ seems more natural than under $\mathrm{IPR_1}$.

[11] It is well known that the same Π^0_2 formulas are provable in intuitionistic and classical number theory. Similarly, the same Π^1_1 statements are provable in $\mathcal{B}$ + MP + $\mathrm{DC}^\bullet_1$ as in its classical equivalent. Since every prenex theorem of $\mathcal{A}$ is provable in its classically correct subtheory by the method of [9], $\mathcal{A}$ and the classical equivalent of $\mathcal{B} + \mathrm{DC_1}$ have the same Π^1_1 theorems.

[12] In informal axiomatic reasoning, the difference between a *rule* ("If A holds, then B holds") and an *axiom* ("If A then B" or "$A \rightarrow B$") is subtle but important. In order to apply a rule, one must already have proved the hypotheses outright on the basis of the declared axioms, using only the declared logical principles; thus a rule which holds for a theory $\mathcal{S}$ need not hold for an extension (by logical or mathematical principles) of $\mathcal{S}$.

not be recursive). With a little help from the Normal Form Theorem, Kleene's Rule implements Church's Thesis by identifying the decidable relations with the recursive relations.

Proposition 3.2. (a) *Every recursive relation is decidable.*
(b) *Every recursive relation is equivalent to its double negation.*
(c) *The class of recursive relations contains all equations and is closed under* $\wedge, \vee, \rightarrow, \neg, \leftrightarrow$.

Proofs: (a) follows from Kleene's Normal Form Theorem (see the development in [10] with [7]). To each recursive relation R on ${}^\omega\omega \times \omega$, for example, there is a numerical code e such that $\forall\alpha\forall x\exists! y T(e, x, \overline{\alpha}(y))$ and

$$\begin{aligned} R(\alpha, x) &\leftrightarrow \exists y[T(e, x, \overline{\alpha}(y)) \wedge U(y) = 1] \\ &\leftrightarrow \forall y[T(e, x, \overline{\alpha}(y)) \rightarrow U(y) = 1] \end{aligned}$$

where χ_T and U are primitive recursive. Given α and x, let $w = \overline{\alpha}(y)$ for the (least) y such that $T(e, x, \overline{\alpha}(y))$; then $R(\alpha, x)$ holds if $U(w) = 1$, and $\neg R(\alpha, x)$ holds if $U(w) \neq 1$. (b) is immediate from (a) by intuitionistic logic.

For (c), if ϕ, ψ are (primitive) recursive, $1 \dot{-} |\phi(\alpha, \beta, x, y) - \psi(\alpha, \gamma, x, z)|$ is a recursive characteristic function of $\phi(\alpha, \beta, x, y) = \psi(\alpha, \gamma, x, z)$. If $R(\alpha, \beta, x, y)$ and $Q(\alpha, \gamma, x, z)$ have recursive characteristic functions χ_R, χ_Q respectively, then $\chi_{R\wedge Q} = \chi_R \cdot \chi_Q$ and $\chi_{\neg R} = 1 \dot{-} \chi_R$. Closure under $\vee$ and $\rightarrow$ follows by (a) with the fact that if A and B are decidable, then $(A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$ and $(A \rightarrow B) \leftrightarrow \neg(A \wedge \neg B)$; and $\leftrightarrow$ is definable in terms of $\rightarrow$ and $\wedge$. $\square$

Definition 3.3. *A relation* $R(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)$ *is* arithmetical *if and only if it is definable from equations using only* $\vee, \wedge, \neg, \rightarrow$ *and number quantifiers.*

Definition 3.4. *The levels* Π^0_n, Σ^0_n *and* Δ^0_n *of the* constructive arithmetical hierarchy *are defined as follows.*[13] *A relation* $R(x, \alpha)$ *is* Π^0_1 *if and only if it is expressible in the form* $\forall y P(y, x, \alpha)$ *where* P *is recursive;* Σ^0_1 *if and only if it can be expressed in the form* $\exists y Q(y, x, \alpha)$ *where* Q *is recursive;* Δ^0_1 *if and only if it is both* Σ^0_1 *and* Π^0_1*. For* $n > 1$*, a relation* $R(x, \alpha)$ *is* Π^0_n *if and only if it can be expressed in the form* $\forall y P(y, x, \alpha)$ *where* P *is* Σ^0_{n-1}*;* $R(x, \alpha)$ *is* Σ^0_n *if and only if it is expressible as* $\exists y Q(y, x, \alpha)$ *where* Q *is* Π^0_{n-1}*; and for all* $n \geq 1$*:*

$$\Delta^0_n = \Pi^0_n \cap \Sigma^0_n.$$

Lemma 3.5. *The following general logical principles hold intuitionistically:*
(a) $\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$ *but only* $\neg A \vee \neg B \rightarrow \neg(A \wedge B)$.
(b) $\neg\neg(A \wedge B) \leftrightarrow \neg\neg A \wedge \neg\neg B$.
(c) $\neg(A \rightarrow B) \leftrightarrow \neg(\neg A \vee B) \leftrightarrow \neg\neg A \wedge \neg B$.
(d) $A \rightarrow \neg\neg A$ *and hence* $\neg\neg\neg A \leftrightarrow \neg A$.
(e) $\neg\neg(A \rightarrow B) \leftrightarrow (A \rightarrow \neg\neg B) \leftrightarrow (\neg\neg A \rightarrow \neg\neg B) \leftrightarrow (\neg B \rightarrow \neg A)$.
(f) $\neg\neg\forall x C(x) \rightarrow \forall x \neg\neg C(x)$.
(g) $\neg\exists x C(x) \leftrightarrow \forall x \neg C(x) \leftrightarrow \neg\neg\forall x \neg C(x)$.
(h) $\exists x \neg C(x) \rightarrow \neg\forall x C(x)$.

In (f),(g),(h)*, the* x *may be replaced by* α.

[13] By coding it suffices to state the definition for $R(x, \alpha)$.

Lemma 3.6. *The following general principles hold intuitionistically, provided that x is not free in B, nor in $D(y)$ (unless y is x).*[14]

(a) $B \vee \exists x C(x) \leftrightarrow \exists x(B \vee C(x))$.
(b) $B \wedge \exists x C(x) \leftrightarrow \exists x(B \wedge C(x))$.
(c) $B \vee \forall x C(x) \rightarrow \forall x(B \vee C(x))$.
(d) $B \wedge \forall x C(x) \leftrightarrow \forall x(B \wedge C(x))$.
(e) $\exists x C(x) \vee \exists y D(y) \leftrightarrow \exists x(C(x) \vee D(x))$.
(f) $\forall x C(x) \wedge \forall y D(y) \leftrightarrow \forall x(C(x) \wedge D(x))$.
(g) $\exists x C(x) \wedge \exists y D(y) \leftrightarrow \exists x(C((x)_0) \wedge D((x)_1))$.
(h) $\forall x C(x) \vee \forall y D(y) \leftrightarrow \exists z \forall x((z = 0 \rightarrow C(x)) \wedge (z > 0 \rightarrow D(x)))$.

Either or both of x, y may be replaced by α, β with the corresponding restrictions.

Proposition 3.7. *Every Δ^0_1 relation is recursive, and conversely, every recursive relation is Δ^0_1.*

Proof: Without loss of generality consider a relation R on ${}^\omega\omega \times \omega$ such that $R(\alpha, x) \leftrightarrow \forall y P(\alpha, x, y)$ and $R(\alpha, x) \leftrightarrow \exists z S(\alpha, x, z)$, where both P and S are recursive. By Proposition 3.2(b) with Lemmas 3.5(g), 3.6(f) and 3.5(a):

$$\begin{aligned} \neg(R(\alpha,x) \wedge \neg R(\alpha,x)) &\leftrightarrow \neg(\forall y P(\alpha,x,y) \wedge \neg\exists z S(\alpha,x,z)) \\ &\leftrightarrow \neg\forall y \neg(\neg P(\alpha,x,y) \vee S(\alpha,x,y)). \end{aligned}$$

Since $\forall\alpha\forall x \neg(R(\alpha,x) \wedge \neg R(\alpha,x))$ holds, and the relation $\neg P(\alpha,x,y) \vee S(\alpha,x,y)$ is decidable by Proposition 3.2(c) and (a), Markov's Rule gives

$$\forall\alpha\forall x\exists y(\neg P(\alpha,x,y) \vee S(\alpha,x,y)).$$

Hence by Kleene's Rule there is a recursive functional φ such that

$$\forall\alpha\forall x(\neg P(\alpha,x,\varphi(\alpha,x)) \vee S(\alpha,x,\varphi(\alpha,x))),$$

and we have $R(\alpha,x) \leftrightarrow S(\alpha,x,\varphi(\alpha,x))$. The converse is trivial. $\square$

Proposition 3.8. *For every $n \geq 1$:*

$$\Pi^0_n \cup \Sigma^0_n \subseteq \Delta^0_{n+1}.$$

Proof: If $R(\alpha, x)$ is in $\Pi^0_n \cup \Sigma^0_n$, then

$$R(\alpha,x) \;\leftrightarrow\; \forall z R(\alpha,x) \;\leftrightarrow\; \exists z R(\alpha,x).$$

Choose the first equivalent if R is Σ^0_n, or the second if R is Π^0_n. $\square$

The constructive arithmetical hierarchy consists of all *positive* arithmetical relations, those definable from recursive relations using only $\wedge, \vee$ and number quantifiers.[15] Even very simple positive arithmetical relations such as

$$R(\alpha) \;\equiv\; \forall x(\alpha(x) = 0) \vee \exists y(\alpha(y) = 3)$$

place higher in this constructive hierarchy than they do in the standard classical hierarchy; see the remark following the proof of Proposition 7.2 below.[16] By

[14]The formal restriction "x is not free in B" translates into the obvious condition on relations. In Lemmas 3.5, 3.6, 3.14, 3.20 and 5.1 it is understood that any variables other than the restricted ones may appear in $A, B, C, \ldots$.

[15]Wim Veldman's [20], [21], [22] study the constructive Borel and projective hierarchies from a strictly intuitionistic point of view.

[16]A strict intuitionist would not accept that argument, which requires Markov's Principle, but he or she would not be able to show that this $R(\alpha)$ is Π^0_2.

Proposition 3.2(b) and (a), Lemma 3.5(a), (g) and (b), and Lemma 3.6(b) and (d), however:

$$\neg R(\alpha) \quad \leftrightarrow \quad \neg\neg\exists x \forall y \neg[(\alpha(x) = 0) \vee (\alpha(y) = 3)]$$

and so by Lemma 3.5(d) and (g):

$$\neg\neg R(\alpha) \quad \leftrightarrow \quad \forall x \neg\neg\exists y[(\alpha(x) = 0) \vee (\alpha(y) = 3)].$$

Another exercise in intuitionistic logic establishes

$$\neg\neg R(\alpha) \quad \leftrightarrow \quad \neg\neg\exists y \forall x[(\alpha(x) = 0) \vee (\alpha(y) = 3)].$$

This suggests the following definitions.

Definition 3.9. *Let $\dot{\exists}$ abbreviate $\neg\neg\exists$. Then $\dot{\exists}x, \dot{\exists}\alpha$ are the* classical existential quantifiers*.*

Definition 3.10. *The levels of the* classical arithmetical hierarchy *are defined as follows. A relation $R(x, \alpha)$ is $\dot{\Pi}^0_1$ if and only if it is expressible in the form $\forall y P(y, x, \alpha)$ where P is recursive; $\dot{\Sigma}^0_1$ if and only if it can be expressed in the form $\dot{\exists} y Q(y, x, \alpha)$ where Q is recursive. For $n > 1$, a relation $R(x, \alpha)$ is $\dot{\Pi}^0_n$ if and only if it can be expressed in the form $\forall y P(y, x, \alpha)$ where P is $\dot{\Sigma}^0_{n-1}$; $\dot{\Sigma}^0_n$ if and only if it is expressible as $\dot{\exists} y Q(y, x, \alpha)$ where Q is $\dot{\Pi}^0_{n-1}$; and for all $n \geq 1$:*

$$\dot{\Delta}^0_n \; = \; \dot{\Pi}^0_n \cap \dot{\Sigma}^0_n.$$

The $\neg\neg R(\alpha)$ in the example above is $\dot{\Delta}^0_2$. While its intuitionistic reduction required some thought, the $\dot{\Pi}^0_2$ and $\dot{\Sigma}^0_2$ equivalents obtained are $\forall x \dot{\exists} y Q$ and $\dot{\exists} y \forall x Q$ with the same recursive relation Q one would naturally get in reducing $R(\alpha)$ by classical logic to $\forall x \exists y Q$ and $\exists y \forall x Q$.

Proposition 3.11. *Every $\dot{\Delta}^0_1$ relation is recursive, and conversely.*

Proof: If $R(\alpha, x) \leftrightarrow \dot{\exists} y S(\alpha, x, y)$ then $\neg R(\alpha, x) \leftrightarrow \neg\exists y S(\alpha, x, y)$; now follow the proof of Proposition 3.7, observing for the converse that $\dot{\exists} z R(\alpha, x) \leftrightarrow \exists z R(\alpha, x)$ (because $\neg\neg R(\alpha, x) \leftrightarrow R(\alpha, x)$). □

Proposition 3.12. *For every $n \geq 1$:*

$$\dot{\Pi}^0_n \cup \dot{\Sigma}^0_n \subseteq \dot{\Delta}^0_{n+1}.$$

Definition 3.13. *A relation $R(\alpha_1, \dots, \alpha_j, x_1, \dots, x_k)$ is* classical arithmetical *if and only if R can be defined from equations using only $\wedge$, $\neg$ and universal number quantifiers $\forall x$.*

Lemma 3.14. *The following principles hold intuitionistically, provided that x is not free in $D(y)$ (unless y is x).*

(a) $\neg\dot{\exists} x C(x) \leftrightarrow \neg\exists x C(x) \leftrightarrow \forall x \neg C(x)$.
(b) $\neg\neg\dot{\exists} x C(x) \leftrightarrow \dot{\exists} x C(x) \leftrightarrow \neg\forall x \neg C(x) \leftrightarrow \dot{\exists} x \neg\neg C(x)$.
(c) $\dot{\exists} x C(x) \vee \dot{\exists} y D(y) \rightarrow \dot{\exists} x (C(x) \vee D(x))$.
(d) $\dot{\exists} x C(x) \; \wedge \; \dot{\exists} y D(y) \leftrightarrow \dot{\exists} x (C((x)_0) \; \wedge \; D((x)_1))$.
(e) $\neg\forall x \dot{\exists} y C(x, y) \leftrightarrow \dot{\exists} x \forall y \neg C(x, y)$.
(f) $\neg\dot{\exists} x \forall y \neg C(x, y) \leftrightarrow \forall x \dot{\exists} y C(x, y)$.
(g) $\dot{\exists} x \dot{\exists} y C(x, y) \leftrightarrow \dot{\exists} x \exists y C(x, y)$.

Either or both of x, y may be replaced by α, β with similar restrictions.

Lemma 3.15. *Every relation belonging to* $\bigcup_{n=1}^{\infty}(\dot\Pi^0_n \cup \dot\Sigma^0_n)$ *is classical; and every classical arithmetical relation* $R(\alpha, x)$ *belongs to some level of the classical arithmetical hierarchy.*

Proof. The first statement holds by Lemma 3.14(b) with the proofs of Prop. 3.2(a) and (c). For the converse, equations are recursive and hence $\dot\Delta^0_1$ by Prop. 3.11. There are three inductive cases.

Case 1. $R(\alpha, x)$ is a conjunction of two relations belonging to the classical arithmetical hierarchy; by Prop. 3.12 we may assume they are both $\dot\Pi^0_m$ for some m. Lemmas 3.6(f) and 3.14(d) reduce $R(\alpha, x)$ inductively to $\dot\Pi^0_m$ form.

Case 2. $R(\alpha, x)$ is the negation of a relation in the classical arithmetical hierarchy. Use Lemma 3.14(a),(b),(e) and (f) with Prop. 3.2(b),(c).

Case 3. $R(\alpha, x)$ is $\forall y S(\alpha, x, y)$ where S belongs to the classical arithmetical hierarchy. By Prop. 3.12 we may assume $S(\alpha, x, y)$ is $\dot\Sigma^0_m$ for some m; then $R(\alpha, x)$ is $\dot\Pi^0_{m+1}$. □

Not all classical relations are (recursively) decidable; an easy but important counterexample is $\forall x\, \alpha(x) = 0$. However, every classical relation is equivalent to its double negation, and this explains the terminology.

Definition 3.16. *A relation* $R(\alpha_1, \dots, \alpha_j, x_1, \dots, x_k)$ *is* stable (under double negation) *if and only if*

$$\forall\alpha_1 \dots \forall\alpha_j \forall x_1 \dots \forall x_k [R(\alpha_1, \dots, \alpha_j, x_1, \dots, x_k) \\ \leftrightarrow \neg\neg R(\alpha_1, \dots, \alpha_j, x_1, \dots, x_k)].$$

Proposition 3.17. *Every classical arithmetical relation is stable.*

Proof. By induction on $n \geq 1$ using Proposition 3.2(b) with Lemma 3.14(b), (e) and (f): every relation in $\dot\Sigma^0_n \cup \dot\Pi^0_n$ is stable. Now use Lemma 3.15. □

Proposition 3.18. *The classical arithmetical hierarchy is proper. That is,*

$$\begin{array}{ccccccccc} & & \dot\Sigma^0_1 & & & & \dot\Sigma^0_2 & & \\ & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\ \dot\Delta^0_1 & & & & \dot\Delta^0_2 & & & & \cdots \\ & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & \\ & & \dot\Pi^0_1 & & & & \dot\Pi^0_2 & & \end{array}$$

Proof: Each level of the classical arithmetical hierarchy contains a complete enumerating relation for that level; just replace each $\exists$ by $\dot\exists$ in the complete relation for the corresponding level of the standard arithmetical hierarchy, given by Kleene's enumeration theorem for the coding of [10]. Adapting the usual diagonal arguments, this hierarchy cannot collapse in a consistent theory, so Proposition 3.12 gives the result. □

To justify a similar statement for the constructive arithmetical hierarchy we need more axioms; cf. §5.

The decision to replace $\exists$ by $\dot\exists$ while retaining $\forall$ may appear arbitrary. According to the Brouwer-Heyting-Kolmogorov interpretation, $\forall x A(x)$ implies that there is a single algorithm which establishes $A(x)$ for *every* x, while

$\neg\neg\forall xA(x)$ and $\forall x\neg\neg A(x)$ are successively weaker constructive assertions. The fact that $\neg\neg\forall\neg\neg$ is intuitionistically equivalent to $\forall\neg\neg$ suggests the following definition.

Definition 3.19. *Let $\dot\forall x$ abbreviate $\forall x\neg\neg$, and let $\dot\forall\alpha$ abbreviate $\forall\alpha\neg\neg$. Then $\dot\forall x$ and $\dot\forall\alpha$ are the* classical universal quantifiers*.*

Proposition 3.20. *The following logical principles hold intuitionistically, as do the corresponding ones with α in place of x and/or β in place of y.*

(a) $\dot\forall x\neg C(x) \leftrightarrow \neg\dot\exists xC(x) \leftrightarrow \forall x\neg C(x) \leftrightarrow \neg\exists xC(x)$.
(b) $\neg\dot\forall x\neg C(x) \leftrightarrow \dot\exists xC(x) \leftrightarrow \neg\forall x\neg C(x)$.
(c) $\neg\dot\exists x\neg C(x) \leftrightarrow \dot\forall xC(x) \leftrightarrow \neg\exists x\neg C(x)$.
(d) $\dot\forall x\dot\exists yC(x,y) \leftrightarrow \forall x\dot\exists yC(x,y) \leftrightarrow \dot\forall x\exists yC(x,y)$.
(e) $\dot\forall x\dot\forall yC(x,y) \leftrightarrow \forall x\dot\forall yC(x,y)$.

Proposition 3.21. *If $A(\alpha,x,y)$ is classical, then $\dot\forall yA(\alpha,x,y) \leftrightarrow \forall yA(\alpha,x,y)$. Hence it makes no difference for the results of this section if each $\forall$ is replaced by $\dot\forall$ in Definitions 3.10 and 3.13.*

4. Axioms of Countable and Dependent Choice

Consider first the countable axiom of numerical choice

$$\mathrm{AC}_0 \qquad \forall x\exists yA(x,y) \to \exists\alpha\forall xA(x,\alpha(x)).$$

The constructive interpretation of $\forall x\exists y$ more than justifies the conclusion, indeed an α must be provided directly by any acceptable verification of the hypothesis. Essentially as Peter Krauss observed in an unpublished manuscript [12] which has received less attention than it deserves, what the classical mathematician *means* by AC_0 can be expressed intuitionistically by

$$\mathrm{AC}_0^\bullet \qquad \dot\forall x\dot\exists yA(x,y) \to \dot\exists\alpha\dot\forall xA(x,\alpha(x)).$$

The $\dot\exists$ in the hypothesis introduces indeterminacy, so that each correlation of a y to an x may be the result of a separate unspecified process; the $\dot\exists$ in the conclusion preserves this indeterminacy while asserting the impossibility (based on the hypothesis) of proving that no α satisfies $\dot\forall xA(x,\alpha(x))$. Either, but not both, of the classical quantifiers in the hypothesis could be replaced by its constructive counterpart without changing the meaning; both quantifiers in the conclusion are essentially classical. Alternatively, $\mathrm{AC}_0^\bullet$ could be stated

$$\mathrm{AAC}_0^\bullet \qquad \forall x\dot\exists y\neg A(x,y) \to \dot\exists\alpha\forall x\neg A(x,\alpha(x)).$$

Beginning with the stronger constructive principle of countable choice

$$\mathrm{AC}_1 \qquad \forall x\exists\alpha A(x,\alpha) \to \exists\beta\forall xA(x,\lambda y.\beta(x,y)),$$

similar considerations justify

$$\mathrm{AC}_1^\bullet \qquad \dot\forall x\dot\exists\alpha A(x,\alpha) \to \dot\exists\beta\dot\forall xA(x,\lambda y.\beta(x,y))$$

and its alternative version $\mathrm{AAC}_1^\bullet$. The converses of all four choice axioms are provable by intuitionistic logic.

Proposition 4.1. (a) AC_0 *is derivable from* AC_1*.*
(b) $\mathrm{AC}_0^\bullet$ *is derivable from* $\mathrm{AC}_1^\bullet$*.*

Proof of (a): Assume $\forall x \exists y A(x,y)$, and define $B(x,\alpha) \equiv A(x,\alpha(0))$. Then $\forall x \exists \alpha B(x,\alpha))$ (for each x, take α to be the constant sequence with value y from the hypothesis of AC_0), so by AC_1 there is a β satisfying $\forall x B(x, \lambda y.\beta(x,y))$, and then $\alpha = \lambda x.\beta(x,0)$ satisfies the conclusion of AC_0. (b) is similar. □

Both $\mathrm{AC}_0^\bullet$ and $\mathrm{AC}_1^\bullet$ follow immediately from their constructive counterparts by Kuroda's Principle for Numbers (or Double Negation Shift)

$$\mathrm{DNS}_0 \qquad \forall x \neg\neg A(x) \to \neg\neg \forall x A(x),$$

whose converse holds in intuitionistic logic. While the corresponding type-1 principle $\forall \alpha \neg\neg A(\alpha) \to \neg\neg\forall\alpha A(\alpha)$ conflicts with continuity, its special case

$$\mathrm{DNS}_1 \qquad \forall \alpha \dot{\exists} x R(\overline{\alpha}(x)) \to \neg\neg \forall \alpha \exists x R(\overline{\alpha}(x))$$

follows from DNS_0 using AC_0 and $\mathrm{MP}_{\mathrm{PR}}$. In the presence of AC_0, DNS_0 is equivalent to the schema obtained by strengthening the $\dot{\forall} x$ in the conclusion of $\mathrm{AC}_0^\bullet$ to $\forall x$.[17]

Although DNS_0 proves its own realizability and q-realizability, it is essentially a logical principle rather than a mathematical one.[18] For this reason we do not treat DNS_0 as an axiom schema, preferring the classical principles of countable and dependent choices which suffice to develop classical second order arithmetic.[19]

Proposition 4.2. *Assuming* AC_1, *every relation* $R(\alpha,x)$ *in the constructive arithmetical hierarchy is expressible in the form* $\exists\beta\forall y A(\alpha,x,\overline{\beta}(y))$ *where* A *is recursive.*

Proof. Since $R \leftrightarrow \exists\beta\forall y R$ for fresh variables β and y, the proposition holds for R recursive. If R is Π^0_1, then for some recursive relation Q:

$$\begin{aligned} R(\alpha,x) &\leftrightarrow \forall y Q(\alpha,x,y) \\ &\leftrightarrow \exists\beta\forall y Q(\alpha,x,lh(\overline{\beta}(y))) \end{aligned}$$

with a recursive length function lh. If R is Σ^0_1, then for some recursive Q:

$$\begin{aligned} R(\alpha,x) &\leftrightarrow \exists z Q(\alpha,x,z) \\ &\leftrightarrow \exists\beta\forall y[lh(\overline{\beta}(y)) > 0 \to Q(\alpha,x,(\overline{\beta}(y))_0 \dot{-} 1)]. \end{aligned}$$

Now suppose R is Σ^0_{n+1} where the proposition holds for Π^0_n. Then by the induction hypothesis there is a recursive relation B such that

$$\begin{aligned} R(\alpha,x) &\leftrightarrow \exists z\exists\gamma\forall y B(\alpha,x,z,\overline{\gamma}(y)) \\ &\leftrightarrow \exists\beta\forall y B(\alpha,x,(\beta(0))_0,\overline{(\beta)_1}(y)) \\ &\leftrightarrow \exists\beta\forall y[lh(\overline{\beta}(y)) > 0 \to C(\alpha,x,\overline{\beta}(y))] \end{aligned}$$

---

[17] If $\forall x\neg\neg A(x)$ then $\forall x \dot{\exists} y[(y=0) \wedge A(x)]$, so $\neg\neg\forall x A(x)$ by the strengthened classical choice schema.

[18] DNS_0 is a discrete countable generalization of the intuitionistically correct $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$. Expanding on the Brouwer-Heyting-Kolmogorov interpretation, one could argue that given a proof, uniform in x, of the impossibility of proving that A(x) is unprovable, then a proof of the impossibility of *ever* proving $\forall x A(x)$ would be an unacceptable limitation on the future development of constructive mathematical practice (which should never, according to Brouwer, be considered as fixed).

[19] $\mathrm{AC}_1^\bullet$ also suffices for Lemma 28 and Theorem 29 of [15]; thus, $\exists\sigma(\sigma\,\mathbf{rf}\,E)$ is provable in the classical version of $\mathbf{B}$, if and only if $\neg\neg E$ is provable in $\mathbf{B} + \mathrm{AC}_1^\bullet + \mathrm{MP}_{\mathrm{PR}} + \mathrm{GC}_1$. Hence DNS_0 and DNS_1 are provable in $\mathcal{A}$.

where $C(\alpha, x, w)$ is $B(\alpha, x, ((w)_0 \dot{-} 1)_0, \overline{\lambda t.((w)_t \dot{-} 1)_1}(lh(w) \dot{-} 1))$.

It remains to show that if the proposition holds for Σ^0_n then it holds for Π^0_{n+1}. Suppose $R(\alpha, x) \leftrightarrow \forall z Q(\alpha, x, z)$ where Q is Σ^0_n. Using AC_1 with the induction hypothesis there is a recursive relation B so that

$$\begin{array}{rcl} R(\alpha, x) & \leftrightarrow & \forall z \exists \gamma \forall y B(\alpha, x, z, \overline{\gamma}(y)) \\ & \leftrightarrow & \exists \beta \forall z \forall y B(\alpha, x, z, \overline{\lambda t.\beta(z, t)}(y)) \\ & \leftrightarrow & \exists \beta \forall y B(\alpha, x, (y)_0, \overline{\lambda t.\beta((y)_0, t)}((y)_1)) \\ & \leftrightarrow & \exists \beta \forall y A(\alpha, x, \overline{\beta}(y)) \end{array}$$

where A is recursive, since the coding satisfies $((y)_0, t) < y$ for all $0 \le t < (y)_1$. $\square$

Proposition 4.3. *Assuming* $\mathrm{AC}^\bullet_1$, *every relation* $R(\alpha, x)$ *in the classical arithmetical hierarchy can be expressed in the equivalent forms* $\dot{\exists} \beta \forall y A(\alpha, x, \overline{\beta}(y))$ *and* $\dot{\exists} \beta \dot{\forall} y A(\alpha, x, \overline{\beta}(y))$ *where* A *is recursive, and also in the equivalent forms* $\forall \beta \dot{\exists} y B(\alpha, x, \overline{\beta}(y))$ *and* $\dot{\forall} \beta \dot{\exists} y B(\alpha, x, \overline{\beta}(y))$ *with a recursive* B.

Proof. The first statement follows by the method of the previous proof, using Proposition 3.2(b) with $\mathrm{AC}^\bullet_1$ and Lemma 3.14(g). For the second statement, observe that $\neg R(\alpha, x)$ can be expressed in the first form by Lemma 3.15. Hence there is a recursive relation $A(\alpha, x, z)$ so that, by Proposition 3.17 with Lemmas 3.14(a)(b), 3.5(g) and Proposition 3.2(b):

$$\begin{array}{rcl} R(\alpha, x) & \leftrightarrow & \neg\neg R(\alpha, x) \\ & \leftrightarrow & \neg \exists \beta \forall y A(\alpha, x, \overline{\beta}(y)) \\ & \leftrightarrow & \forall \beta \neg \forall y \neg\neg A(\alpha, x, \overline{\beta}(y)) \\ & \leftrightarrow & \forall \beta \dot{\exists} y \neg A(\alpha, x, \overline{\beta}(y)) \end{array}$$

where $\neg A(\alpha, x, z)$ is recursive by Proposition 3.2(c). $\square$

The constructive axiom of dependent numerical choices

DC_0 $$\forall x \exists y A(x, y) \to \exists \alpha \forall x A(\alpha(x), \alpha(x+1))$$

is equivalent to AC_0 using induction with the coding of finite sequences. Its classical counterpart

$\mathrm{DC}^\bullet_0$ $$\dot{\forall} x \dot{\exists} y A(x, y) \to \dot{\exists} \alpha \dot{\forall} x A(\alpha(x), \alpha(x+1))$$

is equivalent to $\mathrm{AC}^\bullet_0$ similarly. More interesting are the constructive axiom of dependent choices in ${}^\omega\omega$,

DC_1 $$\forall \alpha \exists \beta A(\alpha, \beta) \to \forall \alpha \exists \gamma [(\gamma)_0 = \alpha \wedge \forall y A((\gamma)_y, (\gamma)_{y+1})]$$

and its classical version

$\mathrm{DC}^\bullet_1$ $$\dot{\forall} \alpha \dot{\exists} \beta A(\alpha, \beta) \to \dot{\forall} \alpha \dot{\exists} \gamma [(\gamma)_0 = \alpha \wedge \dot{\forall} y A((\gamma)_y, (\gamma)_{y+1})],$$

which expresses part of the force of classical uncountable choice.

Each of the four dependent choice axioms can be relativized to an arbitrary hypothesis, e.g.

RDC_0 $$\begin{array}{l} \forall x [A(x) \to \exists y (A(y) \wedge B(x, y))] \to \\ \quad \forall x [A(x) \to \exists \alpha [\alpha(0) = x \wedge \forall y (A(\alpha(y)) \wedge B(\alpha(y), \alpha(y+1)))]], \end{array}$$

$$\mathrm{RDC}_1 \qquad \forall\alpha[A(\alpha) \to \exists\beta(A(\beta) \wedge B(\alpha,\beta))] \to$$
$$\forall\alpha[A(\alpha) \to \exists\gamma[(\gamma)_0 = \alpha \wedge \forall y(A((\alpha)_y) \wedge B((\alpha)_y,(\alpha)_{y+1}))]].$$

The classical counterparts $\mathrm{RDC}_0^\bullet$, $\mathrm{RDC}_1^\bullet$ replace each $\exists$ by $\dot{\exists}$, and (at least) the $\forall y$ by $\dot{\forall} y$. In the context of $\mathcal{B}$ + MP each proves its own realizability and q-realizability, so each is compatible with $\mathcal{T}^+$.

Proposition 4.4. (a) AC_1 *is derivable from* DC_1.
(b) $\mathrm{AC}_1^\bullet$ *is derivable from* $\mathrm{DC}_1^\bullet$.
(c) DC_0 *is derivable from* RDC_0.
(d) DC_1 *is derivable from* RDC_1.

Proof of (a): Assume $\forall x\exists\beta A(x,\beta)$. Then $\forall\alpha\exists\beta C(\alpha(0),\beta)$ where

$$C(x,\beta) \equiv [\beta(0) = x+1 \,\wedge\, A(x,\lambda t.\beta(t+1))].$$

By DC_1,

$$\forall\alpha\exists\gamma[(\gamma)_0 = \alpha \,\wedge\, \forall x C((\gamma)_x(0),(\gamma)_{x+1})],$$

so (setting $\alpha = \lambda t.0$) there is a γ such that

$$(\gamma)_0 = \lambda t.0 \,\wedge\, \forall x[(\gamma)_{x+1}(0) = (\gamma)_x(0)+1 \,\wedge A((\gamma)_x(0),\lambda y.(\gamma)_{x+1}(y+1))].$$

By mathematical induction, $\forall x(\gamma)_x(0) = x$. Define $\beta = \lambda t.(\gamma)_{(t)_0+1}((t)_1+1))$ (using $\mathrm{AC}_0!$). Then $\forall x A(x,\lambda y.\beta(x,y))$, satisfying the conclusion of AC_1. The proof of (b) is similar, with appropriate double negations; and (c),(d) need no comment. As far as we know, the converse to each part is an open question for $\mathcal{M}$; see the next section for further relevant connections. $\square$

Now we turn to bar induction and continuous choice, the specific contributions of intuitionistic analysis. As we shall see, a semi-classical approach to bar induction has powerful consequences.

5. Bar Induction: Variations and Applications

Brouwer's analysis of the structure of ${}^\omega\omega$ led him to accept a form of induction up to any countable ordinal. The Kleene-Brouwer ordering effectively associates with each tree on ω, each of whose branches is effectively finite (or "barred"), a well-ordering of countable length. Any proof by bar induction over the tree can be recast as a proof by induction along the countable well-ordering. Infinite ordinals never explicitly enter the picture, and yet by means of truncated trees (or "stumps") Brouwer was able to work effectively within Cantor's second number class.

Effective bar induction works like this. Given a relation $R(w)$ such that $\forall\alpha\exists!xR(\overline{\alpha}(x))$, and a property $A(w)$ of sequence numbers satisfying
(i) $\forall w(R(w) \to A(w))$, and
(ii) A "propagates back ... across the junctions" ([11] p. 48) in the sense that if $\forall s A(w * \langle s\rangle)$ then $A(w)$,

then A holds at the root $\langle\,\rangle$, and at every node w which is "not past secured" with respect to the "bar" R (that is, every w with no proper initial segment satisfying R). This form of bar induction is expressed by Kleene's axiom

$$\mathrm{BI!} \qquad \forall\alpha\exists!xR(\overline{\alpha}(x)) \,\wedge\, \forall w(Seq(w) \,\wedge R(w) \,\to A(w))$$
$$\wedge\forall w(Seq(w) \,\wedge\, \forall s A(w * \langle s\rangle) \to A(w)) \;\to\; A(\langle\rangle).$$

As always, parameters of both types are allowed. The $\exists!x$ cannot be weakened to $\exists x$ without conflicting with Brouwer's principle of continuous choice.

Alternative versions, requiring the bar to be decidable or monotone rather than thin, are discussed in [11]; monotone bar induction is stronger in the absence of continuity assumptions. One alternative assumes the bar is determined explicitly by a choice sequence:

$$\mathrm{BI}_1 \qquad \forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0 \wedge \forall w(Seq(w) \wedge \rho(w) = 0 \rightarrow A(w)) \\ \wedge \forall w(Seq(w) \wedge \forall sA(w * \langle s\rangle) \rightarrow A(w)) \rightarrow A(\langle\rangle).$$

The proof that BI_1 is equivalent to BI! by AC_0! is an exercise for the reader. Note that AC_0! asserts the constructive existence of every number-theoretic function effectively definable in the two-sorted language; in this sense our minimal theory $\mathcal{M}$ goes far beyond arithmetical comprehension. The bar axiom BI! insures that ${}^{\omega}\omega$ does not consist entirely of recursive functions ([11] Lemma 9.8), even though (by Kleene's Rule) only recursive functions can be proved outright to exist constructively.

To further confuse the issue, $\mathcal{FIM}$ (which includes BI!) is consistent with there being no non-recursive number-theoretic functions ([14]); thus if a sentence of the form $\dot{\exists}\sigma E(\sigma)$ is provable in $\mathcal{FIM}$, then $\dot{\exists}\sigma[GR(\sigma) \wedge E(\sigma)]$ is consistent with $\mathcal{FIM}$ (where $GR(\sigma)$ expresses "σ is general recursive"). In order to faithfully represent (in the intuitionistic context) *all* classically definable number-theoretic functions we need more axioms, which must be classically correct but consistent with the principles of intuitionistic analysis. Consider, for example, stronger versions of the bar axiom.

The strongest classical version of BI! consistent with continuity is

$$\mathrm{BI}^{\bullet} \qquad \forall\alpha\dot{\exists}xR(\overline{\alpha}(x)) \wedge \forall w(Seq(w) \wedge R(w) \rightarrow \neg A(w)) \\ \wedge \forall w(Seq(w) \wedge \forall s\neg A(w * \langle s\rangle) \rightarrow \neg A(w)) \rightarrow \neg A(\langle\,\rangle),$$

or equivalently

$$\mathrm{ABI}^{\bullet} \qquad A(\langle\,\rangle) \wedge \forall w(Seq(w) \wedge A(w) \rightarrow \dot{\exists}sA(w * \langle s\rangle)) \\ \wedge \forall w(Seq(w) \wedge A(w) \rightarrow \neg R(w)) \rightarrow \dot{\exists}\alpha\forall x\neg R(\overline{\alpha}(x)).$$

The strongest classical version of BI_1 consistent with continuity is

$$\mathrm{BI}_1^{\bullet} \qquad \forall\alpha\dot{\exists}x\rho(\overline{\alpha}(x)) = 0 \wedge \forall w(Seq(w) \wedge \rho(w) = 0 \rightarrow A(w)) \\ \wedge \forall w(Seq(w) \wedge \forall sA(w * \langle s\rangle) \rightarrow A(w)) \rightarrow A(\langle\,\rangle).$$

The first two are classical choice principles in disguise, while the third combines MP_1 with BI_1 and best expresses classical bar induction.

Lemma 5.1. *The following logical principles hold intuitionistically, provided that x is not free in A or $\exists yC(y)$:*

(a) $\forall x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \forall xB(x))$.
(b) $\exists x(A \rightarrow B(x)) \rightarrow (A \rightarrow \exists xB(x))$.
(c) $\dot{\exists}x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \dot{\exists}xB(x))$.
(d) $\forall x(B(x) \rightarrow A) \leftrightarrow (\exists xB(x) \rightarrow A)$.
(e) $\exists x(B(x) \rightarrow A) \rightarrow (\forall xB(x) \rightarrow A)$.
(f) $\dot{\exists}xB(x) \wedge \forall x(B(x) \rightarrow \dot{\exists}yC(y)) \rightarrow \dot{\exists}yC(y)$.

The corresponding principles with α, β instead of x, y hold with similar restrictions. In addition, for numbers only:

(g) $\exists! x B(x) \to \forall x(B(x) \vee \neg B(x))$.
(h) $\exists x B(x) \wedge \forall x(B(x) \vee \neg B(x)) \to \exists! x[B(x) \wedge \forall y(y < x \to \neg B(y))]$.
(i) $\dot{\exists} x B(x) \leftrightarrow \dot{\exists}! x[B(x) \wedge \forall y(y < x \to \neg B(y))]$.

Proofs. (a), (b), (d), (e) and (f) are straightforward. For the backward direction of (c) assume $A \to \dot{\exists} x B(x)$ and (toward a contradiction) $\neg\exists x(A \to B(x))$; then $\forall x \neg(A \to B(x))$ and so $\forall x(\neg\neg A \wedge \neg B(x))$, hence $\neg\neg A$ and $\forall x \neg B(x)$. But by the main hypothesis, $\neg\neg A \to \neg\forall x \neg B(x)$. Contradiction. The forward direction follows from (b) by double negation with Lemma 3.5(e). We leave (g), (h) and (i) as exercises. $\square$

Proposition 5.2. (a) $\mathrm{DC}_1^\bullet$ *is derivable from* $\mathrm{RDC}_1^\bullet$*, and conversely.*
(b) $\mathrm{BI}^\bullet$, $\mathrm{ABI}^\bullet$, $\mathrm{AC}_0^\bullet$, $\mathrm{DC}_0^\bullet$ *and* $\mathrm{RDC}_0^\bullet$ *are interderivable.*
(c) $\mathrm{BI}_1^\bullet$ *is equivalent to* $\mathrm{BI}_1 + \mathrm{MP}_1$.

Proofs. Lemma 5.1(c) establishes the equivalence of $\mathrm{RDC}_j^\bullet$ with $\mathrm{DC}_j^\bullet$, for $j = 0, 1$. The equivalence of $\mathrm{AC}_0^\bullet$ with $\mathrm{DC}_0^\bullet$ is an exercise in coding finite sequences. We prove that $\mathrm{BI}^\bullet$ is derivable from $\mathrm{RDC}_0^\bullet$, and $\mathrm{AC}_0^\bullet$ from $\mathrm{ABI}^\bullet$.

Assume the hypotheses of $\mathrm{BI}^\bullet$, and for contradiction assume $A(\langle\,\rangle)$. Then $\forall w(Seq(w) \wedge A(w) \to \dot{\exists} s A(w * \langle s \rangle))$, and so by $\mathrm{RDC}_0^\bullet$:

$$\dot{\exists}\beta\forall x[Seq(\beta(x)) \wedge A(\beta(x)) \wedge \beta(x+1) = \beta(x) * \langle(\beta(x+1))_x \dot{-} 1\rangle].$$

Given such a β, if $\alpha = \lambda x\, (\beta(x{+}1))_x \dot{-} 1$ then $\forall x A(\overline{\alpha}(x))$ and hence $\forall x \neg R(\overline{\alpha}(x))$ by the second hypothesis of $\mathrm{BI}^\bullet$. Then $\dot{\exists}\alpha\forall x \neg R(\overline{\alpha}(x))$ by Lemma 5.1(f), contradicting the first hypothesis. Hence $\neg A(\langle\,\rangle)$.

To derive $\mathrm{AC}^\bullet$ using $\mathrm{ABI}^\bullet$, assume $\forall x \dot{\exists} y B(x, y)$ and let

$$R(w) \equiv \exists y < lh(w) \neg B(y, (w)_y \dot{-} 1),$$
$$A(w) \equiv \forall y < lh(w) B(y, (w)_y \dot{-} 1).$$

The hypotheses of $\mathrm{ABI}^\bullet$ hold, and the conclusion implies $\dot{\exists}\alpha\dot{\forall} x B(x, \alpha(x))$.

To show that $\mathrm{BI}_1^\bullet$ entails MP_1, assume $\dot{\exists} x \beta(x) = 0$. By $\mathrm{AC}_0!$ there is a ρ such that $\rho(w) = 0 \leftrightarrow Seq(w) \wedge \beta(lh(w)) = 0$. Now apply $\mathrm{BI}_1^\bullet$ with $A(w) \equiv Seq(w) \wedge \exists x \beta(x) = 0$. $\square$

The next theorem follows from a recent result of Solovay, on the negative interpretability in $\mathbf{B} + \mathrm{MP}_1$ of a theory with classical logic, BI_1, primitive recursive functions, and an axiom ensuring that ${}^\omega\omega$ is closed under the Turing jump. In the current context his key lemma reads as follows.

Lemma 5.3 (Solovay). *Assuming* $\mathrm{BI}_1^\bullet$,

$$\forall\alpha\dot{\exists}\zeta\forall x[\zeta(x) = 0 \leftrightarrow \exists y T(x, x, \overline{\alpha}(y)).$$

Rather than simply assuming Solovay's lemma, we illustrate the use of $\mathrm{BI}_1^\bullet$ with intuitionistic logic by adapting his argument to prove Lemma 5.5(c) below. Theorem 5.6 is an immediate consequence of Lemma 5.5 with the decidability of equations. The corollaries which follow are of independent interest.

Definition 5.4. *A relation* $A(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)$ *is* classically decidable with respect to $x_1, \ldots, x_k$ *if and only if*

$$\neg\neg\forall x_1 \ldots \forall x_k[A(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m) \vee \neg A(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)].$$

Lemma 5.5. (a) *Classical decidability is equivalent to the classical existence of a characteristic function. That is,* $A(x_1,\dots,x_k,\alpha)$ *is classically decidable with respect to* $x_1,\dots,x_k$ *if and only if*

$$\dot{\exists}\zeta\forall x_1\dots\forall x_k[\zeta((x_1,\dots,x_k))=0\leftrightarrow A(x_1,\dots,x_k,\alpha)],$$

if and only if

$$\dot{\exists}\zeta\forall x[\zeta(x)=0\leftrightarrow A((x)_0,\dots,(x)_{k\,\dot{-}\,1},\alpha)].$$

(Similarly for $\alpha_1,\dots,\alpha_m$ *in place of* α*.)*

(b) *If* $A(x,y,\alpha)$ *and* $B(x,z,\alpha)$ *are classically decidable with respect to* x,y *and* x,z *respectively, then* $\neg A(x,y,\alpha)$ *is classically decidable with respect to* x,y*, and each of* $A(x,y,\alpha)\wedge B(x,z,\alpha)$*,* $A(x,y,\alpha)\vee B(x,z,\alpha)$ *and* $A(x,y,\alpha)\to B(x,z,\alpha)$ *is classically decidable with respect to* x,y,z*.*

(c) *Assuming* $\mathrm{BI}_1^\bullet$*: if* $B(x,y,\alpha)$ *is classically decidable with respect to* x,y *then* $\forall yB(x,y,\alpha)$*,* $\dot{\exists}yB(x,y,\alpha)$ *and* $\exists yB(x,y,\alpha)$ *are classically decidable with respect to* x*.*

Proofs. (a) follows essentially from $\mathrm{AC}_0!$ with the decidability of equations. (b) is an exercise in using (a) with $\mathrm{AC}_0!$ and Lemma 5.1(f) (with the proof of Prop. 3.2(c)).

For (c), adapting Solovay's proof, assume ("for $\dot{\exists}$-elimination" as Lemma 5.1(f) suggests) that for some β not free in $B(x,y,\alpha)$,

$$\forall x\forall y[\beta((x,y))=0\leftrightarrow B(x,y,\alpha)].$$

Our aim is to show $\dot{\exists}\gamma\forall x[\gamma(x)=0\leftrightarrow\forall y\beta((x,y))=0]$. Equivalently, using the logical lemmas with the decidability of equations, we shall derive a contradiction from the assumption ($\star$):

$$\forall\gamma\dot{\exists}x[(\gamma((x)_0)=0\wedge\beta(((x)_0,(x)_1))>0)\ \vee\ (\gamma((x)_0)>0\wedge\forall z\beta(((x)_0,z))=0)].$$

Define $\rho(w)=0$ if $Seq(w)$ and for some $j<lh(w)$, either

(i) $(w)_j=1$ and $\exists y<lh(w)\,\beta((j,y))>0$, or
(ii) $(w)_j>1$ and $\beta((j,(w)_j\dot{-}1))=0\vee\exists y<(w)_j\dot{-}1\,\beta((j,y))>0$,

otherwise $\rho(w)=1$. Define

$$\begin{aligned}A(w)\ \equiv\ &\exists j<lh(w)[((w)_j=1\wedge\exists y\,\beta((j,y))>0)\\ &\vee((w)_j>1\wedge[\beta((j,(w)_j\dot{-}1))>0\to\exists y<(w)_j\dot{-}1\,\beta((j,y))>0])].\end{aligned}$$

The assumption ($\star$) implies $\forall\gamma\dot{\exists}x\rho(\overline{\gamma}(x))=0$. If w is a sequence number then $\rho(w)=0\to A(w)$. And if $Seq(w)\wedge\forall sA(w*\langle s\rangle)$ but $\neg A(w)$, setting $s=0$ gives $\exists y\,\beta((lh(w),y))>0$, but setting $s>0$ and $z=s\dot{-}1$ we have $\forall z[\beta((lh(w),z))>0\to\exists y<z\,\beta((lh(w),y))>0]$, violating the (constructive) least number principle (Lemma 5.1(h)). Since $\neg\neg A(w)\to A(w)$ by MP_1, $A(\langle\,\rangle)$ follows by $\mathrm{BI}_1^\bullet$; but $A(\langle\,\rangle)$ is impossible.

The classical decidability of $\dot{\exists}yB(x,y,\alpha)$ with respect to x follows by parts (a),(b) with Lemma 3.14(b). If β is a characteristic function of $B(x,y,\alpha)$ with respect to x,y, and δ of $\dot{\exists}yB(x,y,\alpha)$ with respect to x, then $\forall x[\delta(x)=0\leftrightarrow\dot{\exists}y\beta((x,y))=0]$ and so by MP_1: $\forall x[\delta(x)=0\leftrightarrow\exists y\beta((x,y))=0]$. It follows by Lemma 5.1(f) that $\exists yB(x,y,\alpha)$ is classically decidable with respect to x. □

Theorem 5.6. *Assuming* $\mathrm{BI}_1^\bullet$*, every arithmetical relation is classically decidable with respect to its number variables.*

Corollary 5.7. *Assuming* $\mathrm{BI}_1^\bullet$*, classical arithmetical reasoning may be used in deriving a classical conclusion. That is, for* $n \geq 1$*: If each of* $A_1(x,\alpha)$, $A_2(x,\zeta_1,\alpha)$, $\ldots$, $A_n(x,\zeta_1,\ldots,\zeta_{n-1},\alpha)$ *is arithmetical,* ζ_j *is not free in* A_j *for any* $1 \leq j \leq n$*, and* $x,\zeta_1,\ldots,\zeta_n$ *are not free in* $B(\alpha)$ *or* $C(\alpha)$*, and if* $B(\alpha) \wedge \forall x[(\zeta_1(x) = 0 \leftrightarrow A_1(x,\alpha)) \wedge \ldots \wedge (\zeta_n(x) = 0 \leftrightarrow A_n(x,\zeta_1,\ldots,\zeta_{n-1},\alpha))]$ *implies* $C(\alpha)$*, then* $B(\alpha) \rightarrow \neg\neg C(\alpha)$*. If* $C(\alpha)$ *is classical, the conclusion may be strengthened to* $B(\alpha) \rightarrow C(\alpha)$*.*

Proof. By the theorem with Lemma 5.5(a), for each $1 \leq j \leq n$:

$$\forall\zeta_1 \ldots \forall\zeta_{j-1}\dot{\exists}\zeta_j\forall x[\zeta_j(x) = 0 \leftrightarrow A_j(x,\zeta_1,\ldots,\zeta_{j-1})].$$

Hence by induction on n using Lemmas 3.6 and 5.1,

$$\dot{\exists}\zeta_1 \ldots \dot{\exists}\zeta_n\forall x[(\zeta_1(x) = 0 \leftrightarrow A_1(x,\alpha)) \wedge \ldots \\ \wedge (\zeta_n(x) = 0 \leftrightarrow A_n(x,\zeta_1,\ldots,\zeta_{n-1},\alpha))],$$

which is equivalent by coding with Lemma 3.14(g) to

$$\dot{\exists}\zeta\forall x[((\zeta)_1(x) = 0 \leftrightarrow A_1(x,\alpha)) \wedge \ldots \\ \wedge ((\zeta)_n(x) = 0 \leftrightarrow A_n(x,(\zeta)_1,\ldots,(\zeta)_{n-1},\alpha))].$$

The result now follows by intuitionistic logic. $\square$

Corollary 5.8. *Assuming* $\mathrm{BI}_1^\bullet$*, the constructive arithmetical hierarchy is proper.*

Proof. By induction on $n > 0$ we show that $\Sigma_n^0 \neq \Delta_n^0$ and $\Pi_n^0 \neq \Delta_n^0$. Proposition 3.8 then guarantees that $\Sigma_n^0, \Pi_n^0 \subsetneq \Delta_{n+1}^0$, since if e.g. $\Delta_{n+1}^0 = \Sigma_n^0$ then $\Pi_n^0 \subseteq \Sigma_n^0$ so $\Sigma_n^0 = \Delta_n^0$, contradicting the argument we are about to give.

Basis. $n = 1$. By Kleene's Normal Form Theorem for partial recursive functionals (cf. [10] §4) there are complete enumerating predicates

$$\begin{aligned} R_1(x,y,\alpha) &\equiv \exists z T(x,y,\overline{\alpha}(z)), \\ P_1(x,y,\alpha) &\equiv \forall z \neg T(x,y,\overline{\alpha}(z)) \end{aligned}$$

for the classes of Σ_1^0 and Π_1^0 relations of y,α respectively. Let $E_1(x,\alpha)$ be the diagonal relation $R_1(x,x,\alpha)$, and assume for contradiction that for some e, $\forall\alpha\forall x(E_1(x) \leftrightarrow P_1(e,x,\alpha))$. Then $E_1(e,\alpha) \leftrightarrow \neg E_1(e,\alpha)$, a contradiction. So $E_1(x,\alpha)$ is Σ_1^0 but not Π_1^0. Similarly, $F_1(x,\alpha) \equiv P_1(x,x,\alpha)$ is Π_1^0 but not Σ_1^0. (So far our argument is within $\mathcal{M}$.)

Second Step. $n = 2$. Again by the Normal Form Theorem, let

$$\begin{aligned} R_2(x,y,\alpha) &\equiv \exists z\forall u \neg T(x,y,z,\overline{\alpha}(u)), \\ P_2(x,y,\alpha) &\equiv \forall z\exists u T(x,y,z,\overline{\alpha}(u)). \end{aligned}$$

Consider $E_2(x,\alpha) \equiv R_2(x,x,\alpha)$. Assume for contradiction that for some e, $\forall\alpha\forall x(E_2(x,\alpha) \leftrightarrow P_2(e,x,\alpha))$. Then $\neg E_2(e,\alpha)$, so $E_2(e,\alpha)$ by $\mathrm{MP}_{\mathrm{PR}}$, an impossibility. Hence E_2^0 is not Π_2^0. Similarly by $\mathrm{MP}_{\mathrm{PR}}$, $F_2(x,\alpha) \equiv P_2(x,x,\alpha)$ is not Σ_2^0.

Induction Step. $n = k+2$. Let $R_n(x,y,\alpha)$ be complete Σ_n^0 and $P_n(x,y,\alpha)$ be complete Π_n^0, so there are a Σ_k^0 relation $C(x,y,z,u,\alpha)$ and a Π_k^0 relation $D(x,y,z,u,\alpha)$ such that

$$\begin{aligned} R_n(x,y,\alpha) &\equiv \exists z\forall u C(x,y,z,u,\alpha), \\ P_n(x,y,\alpha) &\equiv \forall z\exists u D(x,y,z,u,\alpha). \end{aligned}$$

Assuming $\mathrm{BI}_1^\bullet$, by Theorem 5.6 with Lemma 5.5:

$$\dot{\exists}\gamma\forall x\forall y\forall z\forall u[\gamma((x,y,z,u))=0 \quad\leftrightarrow\quad C(x,y,z,u,\alpha)],$$
$$\dot{\exists}\delta\forall x\forall y\forall z\forall u[\delta((x,y,z,u))=0 \quad\leftrightarrow\quad D(x,y,z,u,\alpha)].$$

Assume (for $\dot{\exists}$-elimination) that γ and δ are such characteristic functions of $C(x,y,z,u,\alpha)$ and $D(x,y,z,u,\alpha)$ with respect to x,y,z,u. Then

$$R_n(x,y,\alpha) \quad\leftrightarrow\quad \exists z\forall u(\gamma((x,y,z,u))=0),$$
$$P_n(x,y,\alpha) \quad\leftrightarrow\quad \forall z\exists u(\delta((x,y,z,u))=0).$$

The Second Step shows that $E_n(x,\alpha)\equiv R_n(x,x,\alpha)$ (which is Σ^0_n by definition) is not Π^0_n, so $\neg\exists e\forall x[E_n(x,\alpha)\leftrightarrow P_n(e,x,\alpha)]$. Similarly, $F_n(x,\alpha)\equiv P_n(x,x,\alpha)$ is not Σ^0_n. Since these conclusions are negative and do not involve γ,δ explicitly, by the previous corollary $E_n(x,\alpha)$ is not Π^0_n and $F_n(x,\alpha)$ is not Σ^0_n. $\square$

Corollary 5.9. *Assuming* $\mathrm{BI}_1^\bullet$*, if* $P(x,\gamma,w)$ *and* $Q(x,\gamma,w)$ *are primitive recursive relations then*

$$\forall x[A(x,\gamma)\leftrightarrow\dot{\exists}\alpha\forall zP(x,\gamma,\overline{\alpha}(z))\leftrightarrow\forall\beta\dot{\exists}z\neg Q(x,\gamma,\overline{\alpha}(z))]$$
$$\rightarrow\neg\neg\forall x(A(x,\gamma)\vee\neg A(x,\gamma)),$$

so every number-theoretic relation which is $\dot{\Delta}^1_1$*-definable (perhaps with choice sequence parameters) has a classical characteristic function.*

Proof. Omitting the parameter γ for readability, our assumption becomes

(1) $$\forall x[A(x)\leftrightarrow\dot{\exists}\alpha\forall zP(x,\overline{\alpha}(z))]\ \wedge\ \forall x[A(x)\leftrightarrow\forall\beta\dot{\exists}z\neg Q(x,\overline{\beta}(z))].$$

From (1) by Lemma 3.14(e) with $\forall x\neg(A(x)\wedge\neg A(x))$,

$$\forall x\neg(\dot{\exists}\alpha\forall zP(x,\overline{\alpha}(z))\wedge\dot{\exists}\beta\forall zQ(x,\overline{\beta}(z))).$$

By Lemma 3.5(b)(d) each $\dot{\exists}$ can be replaced by $\exists$, so by Lemma 3.6(b)(f),

$$\forall x\neg\exists\alpha\exists\beta\forall z[P(x,\overline{\alpha}(z))\wedge Q(x,\overline{\beta}(z))]$$

and then by Prop. 3.2(b),

(2) $$\forall x\forall\alpha\forall\beta\dot{\exists}z[\neg P(x,\overline{\alpha}(z))\vee\neg Q(x,\overline{\beta}(z))].$$

Define

$$P_1(x,u) \quad\equiv\quad \forall n<lh(u)P(x,((u)_0,\dots,(u)_{n\dot{-}1})),$$
$$Q_1(x,v) \quad\equiv\quad \forall n<lh(v)Q(x,((v)_0,\dots,(v)_{n\dot{-}1})),$$

$$S(z)\ \equiv\ z=((z)_0,(z)_1,(z)_2)\wedge Seq((z)_1)\wedge Seq((z)_2)\wedge lh((z)_1)=lh((z)_2).$$

We assume $\mathrm{BI}_1^\bullet$ with its consequences MP_1 and Corollary 5.7. If u,v,w,z are sequence numbers with $lh(u)=lh(v)$ and $lh(w)=lh(z)$, let $(u,v)\succ(w,z)$ abbreviate $lh(u)<lh(w)\wedge\forall k<lh(u)[(u)_k=(w)_k\wedge(v)_k=(z)_k]$. Call (u,v) *good for* x if for each $m>lh(u)$ there exist u',v' such that $lh(u')=lh(v')=m$, $(u,v)\succ(u',v')$, and $P_1(x,u')\vee Q_1(x,v')$.

Since being good for x is an arithmetical property, we may assume for $\dot{\exists}$-elimination that

(3) $$\forall z[\zeta(z)=0\leftrightarrow S(z)\wedge[((z)_1,(z)_2)\ \text{is good for}\ (z)_0]].$$

Fact 1: For every x and n there are u, v with $\zeta((x, u, v)) = 0 \wedge lh(u) = n$. If for some x, n there were no such u, v, then for every α and β: $\neg\forall y P(x, \overline{\alpha}(y))$ and $\neg\forall z Q(x, \overline{\beta}(z))$, whence $\neg A(x) \wedge A(x)$ by (1), a contradiction. Now use MP_1.

Hence for each x, n there is a least (in the natural ordering of integer codes for pairs) (u, v) such that $lh(u) = lh(v) = n$ and (u, v) is good for x. Let $\eta(x, n)$ (abbreviating $\eta((x, n))$) be this *n-best pair for x*, and for each x, n, k define

$$(4) \quad \xi(x, n, k) = \begin{cases} \eta(x, k) & \text{if } k \geq n, \\ (u, v) & \text{if } k < n \text{ and } (u, v) \succ \eta(x, n) \text{ and } lh(u) = k. \end{cases}$$

Fact 2: For every x, k the sequence $\xi(x, 0, k), \xi(x, 1, k), \xi(x, 2, k), \ldots$ is non-decreasing and has a classical upper bound: $\dot{\exists} m \forall n \xi(x, n, k) \leq m$. Moreover, each $\xi(x, n, k)$ is good for x. To see that $\xi(x, n, k)$ is good for x and $\xi(x, n, k) \leq \xi(x, n{+}1, k)$ observe that if (u, v) is good for x, and $(u', v') \succ (u, v)$, then (u', v') is good for x. For boundedness, assume $\forall m \dot{\exists} n \xi(x, n, k) > m$. Suppose α, β are such that $\forall y P_1(x, \overline{\alpha}(y)) \vee \forall z Q_1(x, \overline{\beta}(z))$; then for every n, $(\overline{\alpha}(n), \overline{\beta}(n))$ is good for x, so $\forall n\, \xi(x, n, k) \leq (\overline{\alpha}(k), \overline{\beta}(k))$, contradicting our assumption. Hence $\forall \alpha \dot{\exists} y \neg P_1(x, \overline{\alpha}(y)) \wedge \forall \beta \dot{\exists} z \neg Q_1(x, \overline{\beta}(z))$, so $\neg A(x) \wedge A(x)$, which is impossible. Hence $\forall x \forall k \dot{\exists} m \forall n \xi(x, n, k) \leq m$.

For $\dot{\exists}$-elimination, assume

$$(5) \qquad \forall x \forall k \forall m [\phi(x, k, m) = 0 \leftrightarrow \forall n \xi(x, n, k) \leq m].$$

By MP_1 with Fact 2, for each x, k there is a least m such that $\phi(x, k, m) = 0$. Denote the least such m by $\psi(x, k)$.

Fact 3: For every x and k, $\dot{\exists} m \forall n \geq m [\xi(x, n, k) = \psi(x, k)]$ and therefore $\psi(x, k) \succ \psi(x, k+1)$. If some x, k failed to satisfy the first statement, then because $\forall n [\xi(x, n, k) \leq \psi(x, k)]$ we would have $\forall m \dot{\exists} n \geq m [\xi(x, n, k) < \psi(x, k)]$, so $\forall n \xi(x, n, k) < \psi(x, k)$ by Fact 2, contradicting the definition of $\psi(x, k)$. The second statement follows from the first by (4). *Note*: $\psi(x, k)$ is good for x.

Given x, let (α, β) be the unique pair such that $\forall k [(\overline{\alpha}(k), \overline{\beta}(k)) = \psi(x, k)]$. By Fact 3, $\forall z [P_1(x, \overline{\alpha}(z)) \vee Q_1(x, \overline{\beta}(z))]$. By (2) with MP_1, there is a y such that $\neg P(x, \overline{\alpha}(y)) \vee \neg Q(x, \overline{\beta}(y))$. If $\neg P(x, \overline{\alpha}(y))$ then $\forall z Q_1(x, \overline{\beta}(z))$ so $\neg A(x)$. If $\neg Q(x, \overline{\beta}(y))$ then $\forall z P_1(x, \overline{\alpha}(z))$ so $A(x)$. In either case, $A(x) \vee \neg A(x)$.

We have proved $\forall x [A(x) \vee \neg A(x)]$ from (1) under the assumption that ζ, η, ξ, ϕ, ψ exist satisfying (3),(4),(5), and so forth. Hence by Corollary 5.7 we have proved that (1) implies $\neg\neg\forall x [A(x) \vee \neg A(x)]$.[20] $\square$

Remark 5.10. Before leaving this subject we should observe that, assuming $\mathrm{AC}_0^\bullet$, every classical analytical relation is classically decidable with respect to its number variables; from this point of view, $\mathrm{BI}_1^\bullet$ is weaker than the weakest classical choice axiom considered here. Assuming AC_0 and DNS_0 (both of which hold in $\mathcal{A}$), *every* analytical relation is classically decidable with respect to its number variables. It follows that no analytically definable sequence is absent from the continuum according to $\mathcal{A}$, even though only the provably recursive sequences are constructively present.

[20] It is an elementary exercise to rewrite (4),(5), etc. in the form required by Corollary 5.7. For example, (4) can be recast as an arithmetical definition of the characteristic function of $\xi(x, n, k) = y$ in terms of ζ, and ϕ, ψ can be treated similarly.

6. Axioms of Continuous Choice and Domains of Continuity

Brouwer's mathematical notoriety rested mostly on his assertion that every total function on ${}^\omega\omega$ is continuous in the Baire space topology. In [11] Kleene stated as an axiom schema a strong principle of continuous choice which he called "Brouwer's Principle for Functions." Kleene's version uses a choice sequence to code both a continuous functional and its modulus of continuity. Following his original notation, we define

$$\{\sigma\}(\alpha) \simeq \sigma(\overline{\alpha}(\mu y\, \sigma(\overline{\alpha}(y)) > 0)) \dot{-} 1.$$

and

$$\{\sigma\}[\alpha] \simeq \lambda x.\sigma(\langle x\rangle * \overline{\alpha}(y_x)) \dot{-} 1 \text{ where } y_x \simeq \mu y\, \sigma(\langle x\rangle * \overline{\alpha}(y)) > 0.$$

Thus $\lambda\alpha.\{\sigma\}(\alpha)$ is a continuous partial function from ${}^\omega\omega$ to ω, defined on sequences α for which $\exists y\sigma(\overline{\alpha}(y)) > 0$; if defined, the value of $\{\sigma\}(\alpha)$ is one less than $\sigma(\overline{\alpha}(y))$ for the *least* such y. Similarly, $\lambda\alpha.\{\sigma\}[\alpha]$ is a continuous partial function from ${}^\omega\omega$ to ${}^\omega\omega$ whose value on α is the partial function whose value on x is one less than $\sigma(\langle x\rangle * \overline{\alpha}(y))$ for the *least* y which gives a positive value to this auxiliary total function of x, α and y, if such a y exists.[21]

As in [10] but using $\downarrow$ instead of Kleene's !, $\{\sigma\}(\alpha)\downarrow$ abbreviates $\exists y\sigma(\overline{\alpha}(y)) > 0$, and $\{\sigma\}[\alpha](x)\downarrow$ abbreviates $\exists y\sigma(\langle x\rangle * \overline{\alpha}(y)) > 0$. By extension, $\{\sigma\}[\alpha]\downarrow$ abbreviates $\forall x\{\sigma\}[\alpha](x)\downarrow$, and $\{\sigma\}[\alpha]\downarrow \wedge A(\{\sigma\}[\alpha])$ abbreviates $\{\sigma\}[\alpha]\downarrow \wedge \forall\beta[\forall x\exists y(\sigma(\langle x\rangle * \overline{\alpha}(y)) = \beta(x)+1 \wedge \forall z(z < y \rightarrow \sigma(\langle x\rangle * \overline{\alpha}(z)) = 0)) \rightarrow A(\beta)]$.

Lemma 6.1. *Assuming* $\mathrm{MP_{PR}}$, $\mathrm{AC_0}$ *and* $\mathrm{DNS_0}$*, the principle* $\mathrm{DNS_1}$ *holds:*

$$\forall\alpha\dot{\exists}xR(\overline{\alpha}(x)) \rightarrow \neg\neg\forall\alpha\exists xR(\overline{\alpha}(x)).$$

Proof. Given R, consider the relation $B(w, y)$ which holds if and only if

(i) $y = 0$ and w codes a sequence such that $R(u)$ holds for some initial segment u of w (possibly w itself); or

(ii) $y > 0$ and $y+1$ codes a sequence such that $R(w*(y+1))$ but $R(u)$ does not hold for any proper initial segment u of $w*(y+1)$.

Assume $(\star)$: $\forall\alpha\dot{\exists}xR(\overline{\alpha}(x))$. Then $\forall w[Seq(w) \rightarrow \dot{\exists}!yB(w, y)]$. (Given w, consider the function α_0 such that $\overline{\alpha_0}(lh(w)) = w$ and $\alpha_0(y) = 0$ for $y > lh(w)$. By hypothesis $\dot{\exists}xR(\overline{\alpha_0}(x))$ and so by Lemma 5.1(i) $\dot{\exists}yB(w, y)$, and if the y exists it must be unique.) By Lemma 5.1(c), $\forall w\dot{\exists}y(Seq(w) \rightarrow B(w, y))$; and by $\mathrm{AC_0}$ with $\mathrm{DNS_0}$: $\dot{\exists}\beta\forall w(Seq(w) \rightarrow B(w, \beta(w)))$ and so $\dot{\exists}\beta\forall\alpha\forall xB(\overline{\alpha}(x), \beta(\overline{\alpha}(x)))$. Hence by $(\star)$: $\dot{\exists}\beta\forall\alpha[\forall xB(\overline{\alpha}(x), \beta(\overline{\alpha}(x))) \wedge \dot{\exists}x\beta(\overline{\alpha}(x)) = 0]$, and by $\mathrm{MP_{PR}}$ the $\dot{\exists}x$ may be replaced by $\exists x$. But then $\dot{\exists}\beta\forall\alpha\exists xB(\overline{\alpha}(x), 0)$ with the $\exists\beta$ superfluous, so by the definition of B at last $\neg\neg\forall\alpha\exists xR(\overline{\alpha}(x))$. □

Proposition 6.2. *Assuming* $\mathrm{MP_{PR}}$*,* $\mathrm{AC_0}$ *and* $\mathrm{DNS_0}$*,*

$$\forall\alpha\dot{\exists}x\exists yA(\overline{\alpha}(x), y) \rightarrow \dot{\exists}\sigma\forall\alpha[\{\sigma\}(\alpha)\downarrow \wedge A(\overline{\alpha}((\{\sigma\}(\alpha))_0), (\{\sigma\}(\alpha))_1)].$$

Proof. By Lemma 6.1 from $\mathrm{BI}^\bullet$ (with Proposition 5.2(b) and the remark following Proposition 4.1). □

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[21] Although "Kleene brackets" are used here with a different meaning than the usual $\{f\}(x) \simeq U(\mu y.T(f, x, y))$, there should be no confusion because the codes are of different types.

Corollary 6.3. *Assuming* $\mathrm{MP_{PR}}$*,* $\mathrm{AC_0}$ *and* $\mathrm{DNS_0}$*, if* $A(\alpha, x)$ *satisfies*

$$\forall\alpha\forall x(A(\alpha,x) \to \dot{\exists} z\forall\beta(\overline{\beta}(z) = \overline{\alpha}(z) \to A(\beta,x))),$$

then $\forall\alpha\dot{\exists} xA(\alpha,x) \to \dot{\exists}\sigma\forall\alpha[\{\sigma\}(\alpha)\downarrow \wedge A(\alpha,\{\sigma\}(\alpha))]$*. Hence every classically pointwise continuous functional has a classical modulus of continuity.*

Now Brouwer's Principle for Functions can be stated in the form[22]

$$\mathrm{CONT_1} \qquad \forall\alpha\exists\beta A(\alpha,\beta) \to \exists\sigma\forall\alpha[\{\sigma\}[\alpha]\downarrow \wedge A(\alpha,\{\sigma\}[\alpha])].$$

The intuitionistic theory $\mathcal{FIM}$ of [11] is $\mathcal{B}$ + $\mathrm{CONT_1}$. By [11] and [10], $\mathcal{FIM}$ is consistent relative to $\mathcal{B}$ and satisfies Kleene's Rule.

Among the many consequences of $\mathrm{CONT_1}$ are

$$\mathrm{CONT_0} \qquad \forall\alpha\exists xA(\alpha,x) \to \exists\sigma\forall\alpha[\{\sigma\}(\alpha)\downarrow \wedge A(\alpha,\{\sigma\}(\alpha))],$$

which implies $\mathrm{CONT_1}!$ (like $\mathrm{CONT_1}$ but with $\exists!\beta$ instead of $\exists\beta$) and "weak continuity"

$$\mathrm{WC_0} \qquad \forall\alpha\exists xA(\alpha,x) \to \forall\alpha\exists x\exists y\forall\beta[\overline{\alpha}(y) = \overline{\beta}(y) \to A(\beta,x)].$$

Weak continuity is equivalent to $\mathrm{CONT_0}$ by monotone bar induction (but not by BI! or $\mathrm{BI_1}$).

Brouwer's principle relativizes to closed subsets of ${}^{\omega}\omega$ called "spreads", but not to arbitrary domains. Troelstra [17] found an optimal relativization which he called "Generalized Continuity"

$$\mathrm{GC_1} \qquad \forall\alpha[A(\alpha) \to \exists\beta B(\alpha,\beta)] \to \exists\sigma\forall\alpha[A(\alpha) \to \{\sigma\}[\alpha]\downarrow \wedge B(\alpha,\{\sigma\}[\alpha])]$$

where $A(\alpha)$ must be expressible in *almost negative* form. Since $\alpha(0) = \alpha(0)$ is almost negative, $\mathrm{CONT_1}$ follows immediately from $\mathrm{GC_1}$.

Definition 6.4 (Kleene)**.** *A relation R is* almost negative *if and only if R can be expressed without* $\vee$*, and without* $\exists$ *except in parts of the forms* $\exists x(s = t)$ *and* $\exists\alpha(s = t)$ *with* s, t *of type 0.*[23]

Theorem 6.5 (Troelstra)**.** $\mathcal{T} = \mathcal{B} + \mathrm{GC_1}$ *is consistent relative to* $\mathcal{B}$*, and is closed under Kleene's Rule.*

To give an idea of the usefulness of the continuity principles, here are some elementary applications. More will appear in the next sections.

Proposition 6.6. $\mathrm{DC_1}$ *is derivable from* $\mathrm{CONT_1}$*.*

Proof. Assume $\forall\alpha\exists\beta A(\alpha,\beta)$. By $\mathrm{CONT_1}$ there is a σ such that $\forall\alpha[\{\sigma\}[\alpha]\downarrow \wedge A(\alpha,\{\sigma\}[\alpha])]$. By mathematical induction, for each α and x there is a unique γ such that

(i) $(\gamma)_0 = \alpha$.
(ii) For each $y < x$: $(\gamma)_{y+1} = \{\sigma\}[(\gamma)_y]$.
(iii) For each $y > x$: $(\gamma)_y = \lambda t.0$.

[22] One could argue that Brouwer's original intention is more accurately expressed by the corresponding statement $\mathrm{CONT_1}!$ with $\forall\alpha\exists!\beta A(\alpha,\beta)$ as hypothesis, where $\exists!$ means "there is exactly one," but the restricted expressibility of our language and the intuitionistic acceptability of countable choice support Kleene's version.

[23] See [8]. Equivalently, by the Normal Form Theorem, R is almost negative if and only if R can be built from Σ^0_1 relations using only $\wedge$, $\to$, $\neg$ and $\forall$.

Hence by $AC_1!$ (with its proof from $AC_0!$ in $\mathcal{M}$), for each α there is a γ satisfying the conclusion $(\gamma)_0 = \alpha \wedge \forall y A((\gamma)_y, (\gamma)_{y+1})$ of DC_1. □

Proposition 6.7. *Assuming* $CONT_0$ *and* MP_{PR}*, Markov's Principle extends to choice sequences:*

$$\forall\alpha(A(\alpha) \vee \neg A(\alpha)) \wedge \neg\forall\alpha\neg A(\alpha) \to \exists\alpha A(\alpha).$$

Proof. Assume $\forall\alpha(A(\alpha) \vee \neg A(\alpha))$. Then by $CONT_0$ there is a σ such that $\forall\alpha[\{\sigma\}(\alpha)\downarrow \wedge (\{\sigma\}(\alpha) = 0 \leftrightarrow A(\alpha))]$. Consider the primitive recursive relation

$$B(\sigma, w) \equiv Seq(w) \wedge \sigma(w) > 0 \wedge \forall x < lh(w)(\sigma(((w)_0, \dots, (w)_{x \dot{-} 1})) = 0).$$

If $\neg\forall\alpha\neg A(\alpha)$ then by $\forall\alpha\exists x B(\sigma, \overline{\alpha}(x))$ we have $\neg\forall w\neg(B(\sigma, w) \wedge \sigma(w) = 1)$, so by MP_{PR} $\exists w(B(\sigma, w) \wedge \sigma(w) = 1)$ and hence $\exists\alpha A(\alpha)$. □

Corollary 6.8. *Assuming* $CONT_0$ *and* MP_{PR}*, the almost negative relations are the classical analytical relations.*

The property of almost negative relations asserted by GC_1 deserves a name, as it has many applications including a mathematical characterization of almost negativity.

Definition 6.9. *A relation* $A(\alpha)$ *is a* domain of continuity *for an axiomatic theory* $\mathcal{S}$ *if and only if, whenever* $\forall\alpha[A(\alpha) \to \exists\beta B(\alpha, \beta)]$ *is provable in* $\mathcal{S}$*, so is* $\exists\sigma\forall\alpha[A(\alpha) \to \{\sigma\}[\alpha]\downarrow \wedge B(\alpha, \{\sigma\}[\alpha])]$.

Proposition 6.10. *There exist arithmetical relations which do not belong to the classical arithmetical hierarchy.*

Proof. Assuming GC_1, every almost negative relation is a domain of continuity. Consider again the example $R(\alpha) \equiv \forall x(\alpha(x) = 0) \vee \exists x(\alpha(x) = 3)$. To see that $R(\alpha)$ cannot be a domain of continuity, define $\psi(\alpha) = 0$ if $\forall x(\alpha(x) = 0)$, and $\psi(\alpha) = 1$ if $\exists x(\alpha(x) = 3)$; then ψ is defined on $R(\alpha)$ but discontinuous at $\lambda t.0$, where $R(\lambda t.0)$.

Since GC_1 is consistent with $\mathcal{M}$ by Theorem 6.5, $R(\alpha)$ is not almost negative, in particular not classical arithmetical so by Lemma 3.15 $R(\alpha)$ does not belong to the classical arithmetical hierarchy. □

Theorem 6.11. *Let* $\mathcal{B}'$ *be any classically correct extension of* $\mathcal{B}$ *such that*

(a) $\mathcal{B}'$ *is closed under Kleene's Rule.*
(b) $\mathcal{T}' = \mathcal{B}' + GC_1$ *is consistent.*
(c) $\mathcal{T}'$ *satisfies Troelstra's Characterization Theorem with respect to* $\mathcal{B}'$.[24]

Then a relation $A(\alpha)$ *is a domain of continuity for* $\mathcal{T}'$ *if and only if* $A(\alpha)$ *is expressible (provably in* $\mathcal{T}'$*) in almost negative form.*

Proof. The proof of Corollary 25 in [15], for $\mathcal{B}$ and $\mathcal{T}$, extends verbatim. The restriction to $A(\alpha)$ with only α free can be removed; the corresponding but simpler proof of Corollary 10, for Church domains in arithmetic, gives an indication of the method. □

[24]See the introduction for a description of this theorem.

7. The Analytical Hierarchies

Definition 7.1. *A relation $R(\alpha, x)$ is Δ^1_0 if and only if it belongs to the constructive arithmetical hierarchy; and $\Pi^1_0 = \Sigma^1_0 = \Delta^1_0$. For $n > 0$, a relation $R(\alpha, x)$ is Π^1_n if and only if it can be expressed in the form $\forall\beta P(\alpha, \beta, x)$ where $P(\alpha, \beta, x)$ is Σ^1_{n-1}; $R(\alpha, x)$ is Σ^1_n if and only if it is expressible as $\exists\beta Q(\alpha, \beta, x)$ where $Q(\alpha, \beta, x)$ is Π^1_{n-1}; and*

$$\Delta^1_n = \Pi^1_n \cap \Sigma^1_n.$$

Proposition 7.2. *Every positive arithmetical relation is Δ^1_0, and conversely.*

Proof. By induction as usual, but first expressing $\vee$ in terms of $\exists y$ because the logical principle $(\forall x A(x) \vee B) \leftrightarrow \forall x(A(x) \vee B)$ (for x not a variable of B) fails intuitionistically. We illustrate by reducing the relation $R(\alpha)$ from the proof of Proposition 6.10 to Σ^0_2 form (modulo pairing):

$$\begin{aligned} R(\alpha) &\leftrightarrow \exists y[(y = 0 \rightarrow \forall x(\alpha(x) = 0)) \wedge (y > 0 \rightarrow \exists z(\alpha(z) = 3))] \\ &\leftrightarrow \exists y[\forall x(y = 0 \rightarrow \alpha(x) = 0) \wedge \exists z(y > 0 \rightarrow \alpha(z) = 3)] \\ &\leftrightarrow \exists y \exists z \forall x[(y = 0 \rightarrow \alpha(x) = 0) \wedge (y > 0 \rightarrow \alpha(z) = 3)]. \end{aligned}$$

The converse is immediate from the definitions. □

To see that the $R(\alpha)$ just considered is not constructive Π^0_2, observe that $\mathrm{MP_{PR}}$ reduces Π^0_2 to $\dot{\Pi}^0_2$; but $R(\alpha)$ is not a domain of continuity. Hence, assuming $\mathrm{MP_{PR}}$ and GC_1, the constructive arithmetical hierarchy differs from the standard arithmetical hierarchy beginning at the second level. For a thorough analysis of the role of disjunction in the constructive Borel hierarchy within the context of $\mathcal{FIM}$, see [22].

Proposition 7.3. *Assuming* $\mathrm{MP_{PR}}$ *and* GC_1*, there exist arithmetical relations which do not belong to the constructive arithmetical hierarchy.*

Proof: Let $Q(\alpha, x)$ be the complete $\dot{\Sigma}^0_2$ relation $\dot{\exists} y \forall z \neg T(x, x, y, \overline{\alpha}(z))$. Then $Q(\alpha, x)$ is not constructive Δ^1_0. If it were, Proposition 4.2 would provide a recursive A such that $Q(\alpha, x) \leftrightarrow \exists\beta\forall y A(\alpha, x, \overline{\beta}(y))$, and hence by GC_1 (since $Q(\alpha, x)$ is almost negative) there would be a σ such that $\forall\alpha\forall x[Q(\alpha, x) \rightarrow \{\sigma\}[\alpha, x] \downarrow \wedge \forall z A(\alpha, x, \overline{\{\sigma\}[\alpha, x]}(z))]$. Then the $\rightarrow$ could be strengthened to $\leftrightarrow$, and by Kleene's Rule the σ would be recursive, so with a little more work $Q(\alpha, x)$ would be constructive Π^0_2 and so by Markov's Principle also $\dot{\Pi}^0_2$. Hence for some e: $Q(\alpha, x) \leftrightarrow \forall y \dot{\exists} z T(e, x, y, \overline{\alpha}(z))$ and by substituting e for x we would have $Q(\alpha, e) \leftrightarrow \neg Q(\alpha, e)$. □

Theorem 7.4 (Wim Veldman)**.** *Assuming* BI!, AC_1 *and* CONT_1*, a relation $R(\alpha, x)$ belongs to the constructive analytical hierarchy if and only if it can be expressed in the form $\exists\sigma\forall\gamma A(\alpha, x, \sigma, \gamma, y)$ where $A(\alpha, x, \sigma, \gamma, y)$ is Δ^1_0. Thus the intuitionistic analytical hierarchy collapses at Σ^1_2, which contains Π^1_2, so $\Delta^1_2 = \Pi^1_2$.*

The first proof is in [20]. Without Brouwer's Principle the constructive analytical hierarchy does not collapse, but (because continuous choice is consistent with constructive analysis) Bishop constructivists cannot prove conclusively that it does not. The *classical analytical hierarchy*, consisting entirely of stable relations, is better behaved.

Definition 7.5. *A relation $R(\alpha, x)$ is $\dot{\Delta}^1_0$ if and only if it belongs to the classical arithmetical hierarchy, so*

$$\dot{\Delta}^1_0 = \bigcup_{n=0}^{\infty} \dot{\Delta}^0_n = \bigcup_{n=0}^{\infty} \dot{\Pi}^0_n = \bigcup_{n=0}^{\infty} \dot{\Sigma}^0_n.$$

Definition 7.6. *A relation $R(\alpha, x)$ is $\dot{\Pi}^1_1$ if and only if it is expressible in the form $\forall\beta P(\alpha,\beta,x)$ where $P(\alpha,\beta,x)$ is $\dot{\Delta}^1_0$; $\dot{\Sigma}^1_1$ if and only if $R(\alpha,x)$ can be expressed in the form $\dot{\exists}\beta Q(\alpha,\beta,x)$ where $Q(\alpha,\beta,x)$ is $\dot{\Delta}^1_0$. For $n > 1$, a relation $R(\alpha,x)$ is $\dot{\Pi}^1_n$ if and only if it can be expressed in the form $\forall\beta P(\alpha,\beta,x)$ where $P(\alpha,\beta,x)$ is $\dot{\Sigma}^1_{n-1}$; $R(\alpha,x)$ is $\dot{\Sigma}^1_n$ if and only if it is expressible as $\dot{\exists}\beta Q(\alpha,\beta,x)$ where $Q(\alpha,\beta,x)$ is $\dot{\Pi}^1_{n-1}$; and*

$$\dot{\Delta}^1_n = \dot{\Pi}^1_n \cap \dot{\Sigma}^1_n.$$

Definition 7.7. *A relation is* classical analytical *if and only if it can be defined from equations using only $\wedge$, $\neg$ and universal quantifiers $\forall\alpha$, $\forall x$.*

Lemma 7.8. *Every relation belonging to $\bigcup_{n=1}^{\infty}(\dot{\Pi}^1_n \cup \dot{\Sigma}^1_n)$ is classical; and conversely, every classical analytical relation $R(\alpha,x)$ belongs to some level of the classical analytical hierarchy.*

Proof. The first statement holds because $\dot{\exists}$ is definable in terms of $\forall$ and $\neg$. For the converse, first show by induction on Definition 7.7 that every classical analytical relation has a mixed classical prenex form in which a sequence of universal quantifiers of both types, separated perhaps by negations or double negations, precedes a quantifier-free relation. Working from the outside toward the quantifier-free part, convert each $\neg\forall\neg$ to $\dot{\exists}$, and then contract adjacent quantifiers of the same kind, for example by $\forall x \forall\alpha A(x,\alpha) \leftrightarrow \forall\alpha A((\alpha(0))_0, (\alpha)_1)$. Finally, raise the type of every classical number quantifier which precedes a classical choice sequence quantifier, using $\forall x A(x) \leftrightarrow \forall\alpha A(\alpha(0))$ and $\dot{\exists} x A(x) \leftrightarrow \dot{\exists}\alpha A(\alpha(0))$. $\square$

Proposition 7.9. *Every classical analytical relation $R(\alpha,x)$ is stable under double negation.*

Proof. From Lemma 7.8 by the method of proof of Proposition 3.17, with $\neg\neg\dot{\forall}\alpha A(\alpha) \leftrightarrow \dot{\forall}\alpha A(\alpha)$ and $\neg\neg\dot{\exists}\alpha A(\alpha) \leftrightarrow \dot{\exists}\alpha A(\alpha)$. $\square$

Proposition 7.10. *Assuming* $\mathrm{AC}^{\bullet}_1$, *every $\dot{\Sigma}^1_1$ relation $R(\alpha,x)$ can be expressed in the form $\dot{\exists}\beta\forall y A(\alpha,x,\overline{\beta}(y))$ where A is recursive; and every $\dot{\Pi}^1_1$ relation $Q(\alpha,x)$ can be expressed in the form $\forall\beta\dot{\exists}y B(\alpha,x,\overline{\beta}(y))$ where B is recursive.*

Proof. If $R(\alpha,x)$ is $\dot{\Sigma}^1_1$ then by Proposition 4.3 with $\mathrm{AC}^{\bullet}_1$ and intuitionistic logic, there is a recursive C such that

$$\begin{aligned} R(\alpha,x) &\leftrightarrow \dot{\exists}\beta\dot{\exists}\gamma\forall y C(\alpha,x,\beta,\overline{\gamma}(y)) \\ &\leftrightarrow \dot{\exists}\beta\forall y C(\alpha,x,(\beta)_0,\overline{(\beta)_1}(y)). \end{aligned}$$

Since C is recursive, by the Normal Form Theorem there is an e such that

$$R(\alpha,x) \leftrightarrow \dot{\exists}\beta\forall y \neg T(e,\alpha,x,\overline{\beta}(y)).$$

The proof for $\dot{\Pi}^1_1$ is similar. $\square$

Proposition 7.11. *For all $n > 0$: $\dot{\Sigma}^1_n \cup \dot{\Pi}^1_n \subseteq \dot{\Delta}^1_{n+1}$, $\dot{\Sigma}^1_n \not\subseteq \dot{\Pi}^1_n$ and $\dot{\Pi}^1_n \not\subseteq \dot{\Sigma}^1_n$. Hence the classical analytical hierarchy is proper:*

$$\begin{array}{ccccccccccccc}
 & & \dot{\Sigma}^1_1 & & & & \dot{\Sigma}^1_2 & & & & \dot{\Sigma}^1_3 & & \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
\dot{\Delta}^1_1 & & & & \dot{\Delta}^1_2 & & & & \dot{\Delta}^1_3 & & & & \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
 & & \dot{\Pi}^1_1 & & & & \dot{\Pi}^1_2 & & & & \dot{\Pi}^1_3 & &
\end{array}$$

Proof. For the inclusion, observe that if α is not among the variables of A, and $A \leftrightarrow \neg\neg A$, then $A \leftrightarrow \forall\alpha(\alpha(0) = \alpha(0) \wedge A) \leftrightarrow \dot{\exists}\alpha(\alpha(0) = \alpha(0) \wedge A)$.

The noninclusions follow from the Normal Form Theorem as for Proposition 3.18. For example, if $R(\alpha, x)$ is the complete Σ^1_1 relation $\exists\beta\forall y\neg T(x, x, \alpha, \overline{\beta}(y))$, then $\neg\neg R(\alpha, x)$ is complete $\dot{\Sigma}^1_1$. If $\neg\neg R(\alpha, x)$ were $\dot{\Pi}^1_1$ then there would be an f so that $\neg\neg R(\alpha, x) \leftrightarrow \forall\beta\dot{\exists}yT(f, x, \alpha, \overline{\beta}(y))$, and then $\neg\neg R(\alpha, f) \leftrightarrow \neg R(\alpha, f)$, a contradiction. So $\dot{\Sigma}^1_1 \not\subseteq \dot{\Pi}^1_1$. $\square$

It is straightforward to prove that $\dot{\Sigma}^1_n$ and $\dot{\Pi}^1_n$ are closed under substitution of classical functionals whose graphs are $\dot{\Sigma}^1_n$. The basic representation theorem for Π^1_1 in terms of the Kleene-Brouwer ordering is proved by constructive bar induction, and assuming $\mathrm{MP_{PR}}$, Proposition 7.10 implies that $\dot{\Pi}^1_1 = \Pi^1_1$. The Suslin-Kleene Theorem is the natural next step.

Chapter 4 of [16], restricted to the first two types, and with countable ordinals represented by well-founded trees, provides a guide for this basic structure theory. Every result whose proof there is entirely constructive should hold here for both the constructive and the classical hierarchies; classical proofs should carry over here in the classical context only. To motivate this routine work, the following characterization of the domains of continuity suggests applications throughout classical analysis.

8. The Theory $\mathcal{A}$

It is useful to think of the nonclassical theory $\mathcal{A}$ as obtained from a classically correct basic theory by adding the principle of generalized continuous choice. The dependencies detailed above and below make possible a slightly more efficient axiomatization of $\mathcal{A}$ directly. First we take the modular approach.

Definition 8.1. *The (classically correct) theory $\mathcal{B}^\bullet$ is the theory determined by either of the axiomatizations*

$$\mathcal{B}^\bullet \;=\; \mathcal{B} + \mathrm{RDC}_1 + \mathrm{MP_{PR}} + \mathrm{DC}^\bullet_1 \;=\; \mathcal{M} + \mathrm{BI}^\bullet_1 + \mathrm{RDC}_1 + \mathrm{DC}^\bullet_1.$$

The nonclassical theory $\mathcal{A}$ is $\mathcal{B}^\bullet$ + GC_1.

Remark 8.2. In $\mathcal{A}$, every principle named above holds, including AC_0, AC_1, DC_1, $\mathrm{AC}^\bullet_0$, $\mathrm{AC}^\bullet_1$, $\mathrm{RDC}^\bullet_1$, DNS_0, DNS_1, $\mathrm{BI}^\bullet$, CONT_0, CONT_1, WC_0, and all three forms of Markov's Principle. In addition GC_1 entails principles GC_0, $\mathrm{GC}_0!$ of generalized continuous choice and comprehension for constructive partial functionals of type ${}^\omega\omega \to \omega$ with classical domains.[25]

[25] The proof that DNS_0 holds in $\mathcal{A}$ is an elementary exercise in realizability, using $\mathrm{AC}^\bullet_1$ with Theorem 8.4 and Corollary 8.5 below.

Theorem 8.3. *The theory $\mathcal{A}$ is consistent relative to $\mathcal{B}^\bullet$. Each of $\mathcal{A}$, $\mathcal{B}^\bullet$ is closed under Kleene's Rule, and $\mathcal{A}$ satisfies Troelstra's Characterization Theorem with respect to $\mathcal{B}^\bullet$.*

Proof. By [15] it is enough to show that $\mathcal{B}^\bullet$ proves the realizability, and $\mathcal{A}$ proves the q-realizability, of the axioms RDC_1 and $\mathrm{DC}_1^\bullet$ of $\mathcal{A}$ which go beyond $\mathcal{T}^+$. Both are straightforward.[26] □

Theorem 8.4. *To every relation $R(\alpha, x)$ there is a classical relation $R^\bullet(\beta, \alpha, x)$ such that in $\mathcal{A}$:*

$$\forall\alpha\forall x[R(\alpha, x) \leftrightarrow \exists\beta R^\bullet(\beta, \alpha, x)].$$

Proof. By GC_1 and Troelstra's characterization theorem for Kleene function-realizability ([17], [15]), to each R there is an almost negative relation $R'(\beta, \alpha, x)$ (expressing "β realizes $R(\alpha, x)$") such that $R(\alpha, x) \leftrightarrow \exists\beta R'(\beta, \alpha, x)$. By $\mathrm{MP}_{\mathrm{PR}}$, GC_1 and Corollary 6.8, R' has an equivalent classical form $R^\bullet$. □

Corollary 8.5. *Let $R(\alpha, x)$ be any analytical relation. In $\mathcal{A}$, the following are equivalent:*

(a) *$R(\alpha, x)$ is a domain of continuity with respect to α.*
(b) *$R(\alpha, x)$ is almost negative.*
(c) *$R(\alpha, x)$ is classical.*
(d) *$R(\alpha, x)$ belongs to some level of the classical analytical hierarchy.*
(e) *$R(\alpha, x)$ is stable under double negation.*

Proofs. In $\mathcal{A}$, every classical analytical relation is almost negative, hence a domain of continuity by GC_1. Every domain of continuity is almost negative by Theorems 6.11 and 8.3; and by $\mathrm{MP}_{\mathrm{PR}}$ with Corollary 6.8, every almost negative analytical relation is classical. Hence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). Lemma 7.8 proves that (c) $\Leftrightarrow$ (d), and (c) $\Rightarrow$ (e) holds by Proposition 7.9. We prove (e) $\Rightarrow$ (c).

Suppose $\forall\alpha\forall x[R(\alpha, x) \leftrightarrow \neg\neg R(\alpha, x)]$, and let $R^\bullet(\beta, \alpha, x)$ be the classical relation given by Theorem 8.4 such that in $\mathcal{A}$: $\forall\alpha\forall x[R(\alpha, x) \leftrightarrow \exists\beta R^\bullet(\beta, \alpha, x)]$. Then $\forall\alpha\forall x[R(\alpha, x) \leftrightarrow \neg\forall\beta\neg R^\bullet(\beta, \alpha, x)]$, where $\neg\forall\beta\neg R^\bullet(\beta, \alpha, x)$ is classical. □

Corollary 8.6. *In $\mathcal{A}$, every relation $R(\alpha, x)$ can be expressed in the form $\exists\beta A(\beta, \alpha, x)$ where A belongs to some level of the classical analytical hierarchy. The resulting total analytical hierarchy is proper in $\mathcal{A}$, as for each $n > 0$:*

$$\Sigma^1(\dot{\Pi}^1_n) \cup \Sigma^1(\dot{\Sigma}^1_n) \subseteq \Sigma^1(\dot{\Pi}^1_{n+1}) \cap \Sigma^1(\dot{\Sigma}^1_{n+1}),$$
$$\Sigma^1(\dot{\Pi}^1_n) \not\subseteq \Sigma^1(\dot{\Sigma}^1_n) \ \textit{ and } \ \Sigma^1(\dot{\Sigma}^1_n) \not\subseteq \Sigma^1(\dot{\Pi}^1_n).$$

Proof. By Theorem 8.4 with Proposition 7.11, only the noninclusions need proof. The arguments we give for $n = 1$ generalize to all $n > 0$. The first noninclusion will be established *absolutely* in the sense that it holds in any consistent extension of $\mathcal{A}$. To prove the second noninclusion we use Kleene's Rule (which holds for $\mathcal{A}$); cf. Remark 2.2.

To show that $\Sigma^1(\dot{\Pi}^1_1) \not\subseteq \Sigma^1(\dot{\Sigma}^1_1)$, let $R(\alpha, x)$ be $\exists\beta\forall\gamma\dot{\exists}y T(x, x, \alpha, \overline{\beta}(y), \overline{\gamma}(y))$, and suppose for contradiction that $\forall\alpha\forall x[R(\alpha, x) \leftrightarrow \exists\delta\dot{\exists}\varepsilon\forall z P(x, \alpha, \delta, \epsilon, z)]$ where $P(x, \alpha, \delta, \epsilon, z)$ is recursive. Then the complete $\dot{\Sigma}^1_2$ relation $\neg\neg R(\alpha, x)$ would be $\dot{\Sigma}^1_1$, which is impossible.

[26] $\Lambda\sigma\Lambda\alpha\Lambda\tau(\Lambda.\rho, (\Lambda.\lambda t.0, \Lambda x.(\phi(x), \psi(x))))$ realizes RDC_1, where ρ, ϕ, ψ are partial recursive functions defined recursively from σ, α, τ; and $\Lambda\sigma.\Lambda\alpha.\lambda t.0$ realizes $\mathrm{DC}_1^\bullet$.

For the reverse noninclusion, let $R(\alpha, x)$ be $\exists\beta\dot{\exists}\gamma\forall y\neg T(x, x, \alpha, \beta, \overline{\gamma}(y))$ and suppose for contradiction that $\mathcal{A}$ proves $\forall\alpha\forall x[R(\alpha, x) \leftrightarrow \exists\delta\forall\varepsilon\dot{\exists}zQ(x, \alpha, \delta, \varepsilon, z)]$ where $Q(x, \alpha, \delta, \varepsilon, z)$ is recursive. Then we have

$$\forall\alpha\forall x\forall\beta[\dot{\exists}\gamma\forall y\neg T(x, x, \alpha, \beta, \overline{\gamma}(y)) \rightarrow \exists\delta\forall\varepsilon\dot{\exists}zQ(x, \alpha, \delta, \varepsilon, z)],$$

and so by GC_1:

$$\exists\sigma\forall\alpha\forall x\forall\beta[\dot{\exists}\gamma\forall y\neg T(x, x, \alpha, \beta, \overline{\gamma}(y)) \\ \rightarrow \{\sigma\}[(\alpha, \beta, \lambda t.x)] \downarrow \wedge \forall\varepsilon\dot{\exists}zQ(x, \alpha, \{\sigma\}[(\alpha, \beta, \lambda t.x)], \varepsilon, z)],$$

where $\{\sigma\}[(\alpha, \beta, \lambda t.x)] \downarrow$ is Π^0_2, hence $\dot{\Pi}^0_2$ by $\mathrm{MP}_{\mathrm{PR}}$, hence $\dot{\Pi}^1_1$; and by Kleene's Rule the σ is recursive. So there is an f such that

$$\forall\alpha\forall x\forall\beta[\forall\varepsilon\dot{\exists}zT(f, x, \alpha, \beta, \overline{\varepsilon}(z)) \\ \leftrightarrow \{\sigma\}[(\alpha, \beta, \lambda t.x)] \downarrow \wedge \forall\varepsilon\dot{\exists}zQ(x, \alpha, \{\sigma\}[(\alpha, \beta, \lambda t.x)], \varepsilon, z)],$$

whence

$$\forall\alpha\forall\beta[\dot{\exists}\gamma\forall y\neg T(f, f, \alpha, \beta, \overline{\gamma}(y)) \rightarrow \forall\gamma\dot{\exists}yT(f, f, \alpha, \beta, \overline{\gamma}(y))].$$

Thus $\forall\alpha\neg\exists\beta\dot{\exists}\gamma\forall y\neg T(f, f, \alpha, \beta, \overline{\gamma}(y))$, that is, $\forall\alpha\neg R(\alpha, f)$. But this implies $\forall\alpha\forall\beta\forall\gamma\dot{\exists}yT(f, f, \alpha, \beta, \overline{\gamma}(y))$, so (by the choice of f) $\forall\alpha\exists\delta\forall\varepsilon\dot{\exists}zQ(f, \alpha, \delta, \varepsilon, z)$ and thus $\forall\alpha R(\alpha, f)$, a contradiction. Hence $\Sigma^1(\dot{\Sigma}^1_1) \not\subseteq \Sigma^1(\dot{\Pi}^1_1)$ in $\mathcal{A}$. $\square$

Corollary 8.7. *$\mathcal{A}$ can be axiomatized directly by*

$$\mathcal{A} \;=\; \mathcal{M} + \mathrm{BI}^\bullet_1 + \mathrm{DC}^\bullet_1 + \mathrm{GC}_1.$$

Proof. RDC_1 is derivable in $\mathcal{M} + \mathrm{BI!} + \mathrm{MP}_{\mathrm{PR}} + \mathrm{DC}^\bullet_1 + \mathrm{GC}_1$, as follows. First, if $A(\alpha)$ is classical then by GC_1: If $\forall\alpha[A(\alpha) \rightarrow \exists\beta(A(\beta) \wedge B(\alpha, \beta))]$ then there is a σ such that $\forall\alpha[A(\alpha) \rightarrow \{\sigma\}[\alpha] \downarrow \wedge (A(\{\sigma\}[\alpha]) \wedge B(\alpha, \{\sigma\}[\alpha]))]$. From this σ we can define recursively a γ for the conclusion of RDC_1.

If $A(\alpha)$ is arbitrary, by Theorem 8.4 there is a classical C such that $A(\alpha) \leftrightarrow \exists\delta C(\delta, \alpha)$. Assume $\forall\alpha[A(\alpha) \rightarrow \exists\beta(A(\beta) \wedge B(\alpha, \beta))]$; then $\forall\alpha\forall\delta[C(\delta, \alpha) \rightarrow \exists\beta\exists\varepsilon(C(\varepsilon, \beta) \wedge B(\alpha, \beta))]$, so by the previous case $\forall\alpha\forall\delta[C(\delta, \alpha) \rightarrow \exists\zeta[(\zeta)_0 = (\alpha, \delta) \wedge \forall x[C(((\zeta)_{x+1})_1, ((\zeta)_{x+1})_0) \wedge B(((\zeta)_x)_0, ((\zeta)_{x+1})_0)]]]$. But then any γ satisfying $(\gamma)_0 = \alpha$ and $\forall x(\gamma)_x = ((\zeta)_x)_0$ satisfies the conclusion of RDC_1. $\square$

The theory $\mathcal{A}$ is a rich axiomatic structure which invites further exploration. For example, writing $\sigma \star \tau$ for the play of a two-person game (of perfect information) generated when I plays by strategy σ and II plays by strategy τ, the classical Axiom of Projective Determinacy can be stated as

$$\mathrm{PD}^\bullet \qquad \forall\sigma\dot{\exists}\tau\forall\alpha(\sigma \star \tau = \alpha \rightarrow \neg A(\alpha)) \rightarrow \dot{\exists}\tau\forall\sigma\forall\alpha(\sigma \star \tau = \alpha \rightarrow \neg A(\alpha)),$$

where $\forall\alpha(\sigma \star \tau = \alpha \rightarrow \neg A(\alpha)) \leftrightarrow \exists\alpha(\sigma \star \tau = \alpha \wedge \neg A(\alpha))$. Restricting the $A(\alpha)$ to $\dot{\Delta}^1_1$ turns this into a statement of classical Borel determinacy $\mathrm{BD}^\bullet$. Each asserts an apparently new property of the quantifier combination $\forall\sigma\dot{\exists}\tau$ combining aspects of classical uncountable choice with a fixed point property. We know $\mathrm{BD}^\bullet$ is classically correct and consistent with $\mathcal{A}$, because it is provable in classical Zermelo-Frankel set theory, hence consistent with $\mathcal{B}^\bullet$; and $\mathcal{B}^\bullet + \mathrm{BD}^\bullet$ proves that $\mathrm{BD}^\bullet$ is realized by $\Lambda\sigma\lambda t\,0$. The usual extension to q-realizability proves that $\mathcal{A} + \mathrm{BD}^\bullet$ satisfies Kleene's Rule. Is $\mathrm{BD}^\bullet$ provable in $\mathcal{A}$? Is $\mathrm{PD}^\bullet$ consistent with $\mathcal{A}$? What more can one say?

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