Constructive Taxonomy

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How can reverse constructive mathematics be unified?

S. Simpson: The goal of **classical reverse mathematics** is to determine which set existence axioms are needed to prove a particular theorem of “ordinary” (classical) mathematics [**CLASS**].

D. Bridges: **Constructive reverse mathematics** asks

1. Which constructive principles are needed to prove particular theorems of Bishop’s constructive mathematics [**BISH**]?
2. Which nonconstructive principles must be added to BISH to prove particular classical theorems?

W. Veldman: **Intuitionistic reverse mathematics** asks which intuitionistic axioms are needed to prove a particular theorem of intuitionistic analysis [**INT**].

(A. S. Troelstra, M. Beeson): **Russian reverse mathematics** asks which theorems of **RUSS** depend on one or both of (Extended) Church’s Thesis (E)CT_0 and Markov’s Principle MP_0: “If a recursive algorithm cannot fail to converge, then it converges?”

The Three Main Varieties of Constructive Mathematics:
INT, RUSS and BISH are all concerned with natural numbers (also coding rationals) and sequences of numbers (also coding reals). All use intuitionistic (not classical) logic and accept full mathematical induction and definition of functions by primitive recursion.

For analysis, **BISH** $\subsetneq$ **CLASS** $\cap$ **INT** $\cap$ **RUSS** but no two of CLASS, INT and RUSS are fully compatible, e.g. the axiomatic form $\forall\alpha\exists e\forall x\,(\alpha(x) = \{e\}(x))$ of Church's Thesis is accepted by RUSS and consistent with BISH, but inconsistent with INT and CLASS. But INT is consistent with $\forall\alpha\neg\neg\exists e\forall x\,(\alpha(x) = \{e\}(x))$, and all three constructive varieties respect **Church's Rule**: *Only general recursive functions can be proved to exist.*

Markov's Principle, which can be thought of as saying that all integers are standard, is accepted by only RUSS and CLASS but consistent with INT and BISH. All four respect **Markov's Rule**.

Classical reasoning can be rendered intuitionistically using double negations, including Krauss' classical quantifiers $\forall\neg\neg$ and $\neg\neg\exists$.

Working Hypothesis: The goal of **reverse constructive analysis** is to determine which *function existence axioms* are needed to prove particular theorems of INT, RUSS and BISH about $\mathbf{N}$, $\mathbf{N}^{\mathbf{N}}$, $2^{\mathbf{N}}$, $\mathbf{R}$, $2^{\mathbf{R}}$, $\mathbf{N}^{\mathbf{R}}$, $\mathbf{R}^{\mathbf{R}}$, $\mathbf{R}^{\mathbf{N}}$, ... *using intuitionistic logic*, **and** which *additional* theorems are provable in *consistent classical extensions* (e.g. using Markov's Principle).

Language, logic and basic axioms: First we need to specify a formal language, with intuitionistic logic and a common core of mathematical axioms built on a primitive recursive foundation. Two general principles expressible in the language may then be called **constructively equivalent** if each can be derived from (instances of) the other using the logic and the common axioms.

RUSS can be formalized in the language of arithmetic, and BISH or INT in a two-sorted language – but only at the cost of arbitrary assumptions about the *representation* of functions from $\mathbf{N}^{\mathbf{N}}$ to $\mathbf{N}$.

Two highly developed formal systems for intuitionistic analysis (Kleene and Vesley's **FIM**, Troelstra's $\mathbf{EL}$ + BI + CC) have been in use for decades as BISH was developing informally. Veldman's **BIM** and Ishihara's $\mathbf{EL}_{\mathrm{ELEM}}$ provide alternative minimal systems. All are two-sorted, with variables for numbers and sequences.

For the common core we choose a three-sorted system $\mathbf{M}_2$ whose restriction $\mathbf{M}_1$ to the two-sorted language is already familiar.

Logic: three-sorted intuitionistic logic with number-theoretic equality. Equality between functions is defined extensionally: $\alpha = \beta$ abbreviates $\forall x(\alpha(x) = \beta(x))$ and $\Phi = \Psi$ abbreviates $\forall \alpha(\Phi(\alpha) = \Psi(\alpha))$. Extensional equality axioms are assumed.

Terms s,t,... (of type 0), and **functors** u,v,... of type 1 and U,V,... of type 2 are defined from the variables and primitive recursive function constants using application and Church's λ. If U and v are functors and s is a term, then for example

- U(v) + v(s) is a term,
- λx.(U(v) + v(x)) is a functor of type 1, and
- $\lambda \alpha$.(U(α) + α(s)) is a functor of type 2.

If t is a term and x a number variable, we write t(x) for t, and t(s) for the result of substituting s for every free occurrence of x in t. Similarly for U(α), U(v). The λ-**conversion axiom schemas** are

- (λx.t(x))(s) = t(s), and
- ($\lambda \alpha$.U(α))(v) = U(v).

Mathematical axioms for the common constructive core:
First consider the familiar two-sorted minimal systems

- **EL** (Troelstra) based on a generous two-sorted primitive recursive Heyting arithmetic $\mathbf{HA}_1$, with full mathematical induction: $A(0) \wedge \forall x(A(x) \to A(x+1)) \to A(x)$ for all $A(x)$.

  EL assumes **quantifier-free countable choice qf-$\mathbf{AC}_{00}$**:

  $$\forall x \exists y A(x, y) \to \exists \alpha \forall x A(x, \alpha(x))$$

  where $A(x, y)$ is quantifier-free and has no free α.

- $\mathbf{M}_1$ (Kleene, Vesley, JRM) based on a frugal two-sorted intuitionistic arithmetic $\mathbf{IA}_1$ with full mathematical induction.

  $\mathbf{M}_1$ assumes **countable function comprehension $\mathbf{AC}_{00}$!**:

  $$\forall x \exists ! y A(x, y) \to \exists \alpha \forall x A(x, \alpha(x))$$

  for every formula $A(x, y)$ with α and x free for y, where $\exists ! y B(y)$ abbreviates $\exists y B(y)$ & $\forall y \forall z (B(y)$ & $B(z) \to y = z)$.

Proposition 1. $\mathbf{M}_1$ proves qf-AC_{00} and

$\mathbf{CF}_0$: $\quad \forall x(A(x) \vee \neg A(x)) \rightarrow \exists \chi \forall x(\chi(x) = 0 \leftrightarrow A(x)),$

where χ is not free in $A(x)$. $\mathbf{M}_1$ also proves

$\mathbf{AC}_{01}\mathbf{!}$: $\quad \forall x \exists ! \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, \lambda y.\beta(x, y)),$

where β, x are free for α in $A(x, \alpha)$ and $\beta(x, y) \equiv \beta(2^x \cdot 3^y)$.

Theorem 2. (G. Vafeiadou)

(a) **EL** does *not* prove CF_0. That is, **EL** cannot prove that every detachable subset of **N** has a characteristic function.

(b) **EL** + CF_0 proves $AC_{00}!$.

Let $\mathbf{EL}^+$ be the definitional extension of **EL** including symbols and defining axioms for the finitely many constants of $\mathbf{M}_1$. Then

(c) $\mathbf{EL}^+$ is a conservative extension of the theory $\mathbf{M}_1^-$ obtained by replacing $AC_{00}!$ by qf-AC_{00}.

(d) $\mathbf{EL}^+$ + CF_0 is a conservative extension of $\mathbf{M}_1$.

The relation of **BIM** to $\mathbf{M}_1$ is a only a little more complicated.

Does countable choice belong in the constructive core? Brouwer and Bishop accepted countable choice but not all constructivists do. Reverse constructive analysis treats it as an *optional* function existence principle. Unlike "unique choice," countable choice has many nonequivalent forms, e.g.:

The maximal classically correct subsystem **B** of Kleene and Vesley's two-sorted system **FIM** for intuitionistic analysis includes an axiom schema of bar induction and replaces $\mathrm{AC}_{00}!$ by **countable choice:**

$\mathbf{AC}_{01}$: $\quad \forall x \exists \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, \lambda y.\beta(x, y))$.

Over $\mathbf{IA}_1$, $\mathbf{AC}_{01}$ is stronger than its consequence $\mathbf{AC}_{00}$: $\forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$.

A curious variation on countable choice, which follows from $\mathbf{AC}_{00}$ and is interderivable with $\mathrm{AC}_{00}! + \neg\neg\, \mathrm{AC}_{00}$ over $\mathbf{IA}_1$, is $\mathbf{AC}_{00}\mathbf{!!}$:

$$\forall x \exists y A(x, y) \;\&\; \forall \alpha \forall \beta [\forall x A(x, \alpha(x)) \;\&\; \forall x A(x, \beta(x)) \rightarrow \alpha = \beta] \rightarrow \exists \alpha \forall x A(x, \alpha(x)).$$

Why "unique choice" belongs in the constructive common core: Kleene's choice of $AC_{00}!$ (rather than qf-AC_{00} or AC_{00} or AC_{01}) as the minimal function existence principle for $\mathbf{M}_1$ allowed him to formalize the theory of recursive functionals using finitely many primitive recursive function and functional constants and to exploit the difference between classical and intuitionistic logic.

- $\mathbf{IA}_1$ + qf-AC_{00} and **EL** have natural classical models in which the type-1 variables range over all general recursive functions.
- $\mathbf{IA}_1 + AC_{00}!$ (i.e. $\mathbf{M}_1$) and $\mathbf{EL} + CF_0$ do not, since with classical logic, CF_0 gives full comprehension for all properties of numbers expressible in the language.
- However, $\mathbf{M}_1$ and $\mathbf{EL} + CF_0$ only prove the existence of general recursive functions.
- Classical logic does not distinguish between AC_{00} and $AC_{00}!$, since if any witness exists, so does the unique least witness.
- However, $\mathbf{M}_1$ does not prove AC_{00}. (S. Weinstein [1979])

AC_{00} can be decomposed into a *bounded choice schema* $\mathbf{BC}_{00}$:

$$\forall x \exists y \leq \beta(x) A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$$

and a *bounding axiom schema* $\mathbf{AB}_{00}$:

$$\forall x \exists y A(x, y) \rightarrow \exists \beta \forall x \exists y \leq \beta(x) A(x, y).$$

Proposition 3. (a) $\mathbf{IA}_1$ + BC_{00} proves CF_0.

(b) $\mathbf{IA}_1$ + AB_{00} proves qf-AC_{00}, so $\mathbf{M}_1 \subseteq \mathbf{IA}_1 + \mathsf{CF}_0 + \mathsf{AB}_{00}$.

(c) $\mathbf{IA}_1 + \mathsf{AB}_{00} + \mathsf{BC}_{00} = \mathbf{IA}_1 + \mathsf{AC}_{00} = \mathbf{M} + \mathsf{AC}_{00}$.

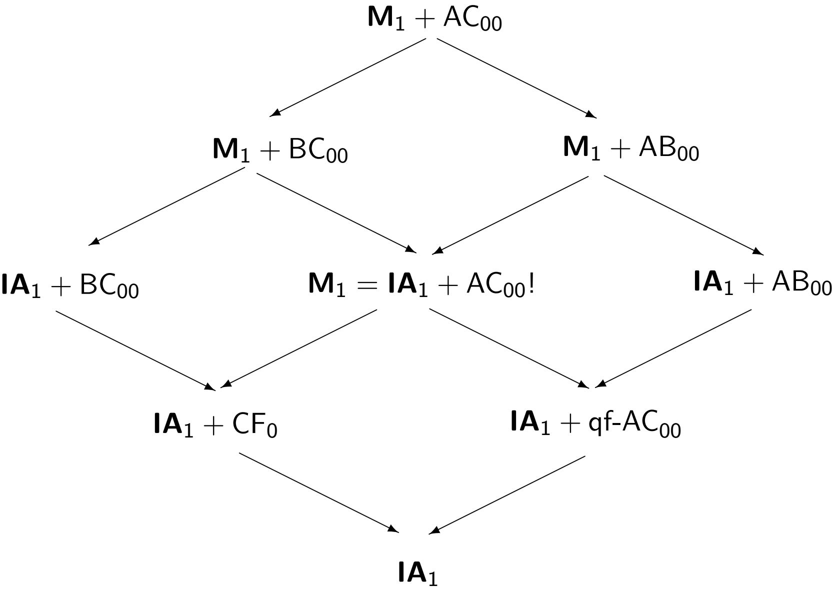
(d) $\mathbf{IA}_1$ + AB_{00} proves that every Cauchy sequence of reals has a modulus of convergence (important for constructive analysis).

Theorem 4. (a) $\mathbf{IA}_1$ + BC_{00} does not prove AB_{00} or qf-AC_{00} (by classical model of primitive recursively bounded sequences).

(b) $\mathbf{M}_1$ does not prove BC_{00} (by Weinstein's Kripke model),

(c) $\mathbf{M}_1$ + AB_{00} does not prove BC_{00} (J. van Oosten, using Lifschitz realizability).

Challenge: Does $\mathbf{M}_1$ + BC_{00} prove AB_{00}?



$\mathbf{M}_1 + \mathrm{AC}_{00}$
$\mathbf{M}_1 + \mathrm{BC}_{00}$
$\mathbf{M}_1 + \mathrm{AB}_{00}$
$\mathbf{IA}_1 + \mathrm{BC}_{00}$
$\mathbf{M}_1 = \mathbf{IA}_1 + \mathrm{AC}_{00}!$
$\mathbf{IA}_1 + \mathrm{AB}_{00}$
$\mathbf{IA}_1 + \mathrm{CF}_0$
$\mathbf{IA}_1 + \text{qf-AC}_{00}$
$\mathbf{IA}_1$

Our **three-sorted minimal theory** $\mathbf{M}_2$, extending $\mathbf{M}_1$, also has a type-2 function comprehension axiom schema

$$\mathbf{AC}_{10}!: \qquad \forall\alpha\exists! m A(\alpha, m) \rightarrow \exists\Phi\forall\alpha A(\alpha, \Phi(\alpha))$$

which entails $\mathrm{AC}_{00}!$ and qf-AC_{10}, guarantees the existence of type-2 general recursive functions and provides a characteristic function for each detachable subset of $\mathbf{N}^{\mathbf{N}}$. That is, $\mathbf{M}_2$ proves

$$\mathbf{CF}_1: \qquad \forall\alpha(A(\alpha) \vee \neg A(\alpha)) \rightarrow \exists\Theta\forall\alpha(\Theta(\alpha) = 0 \leftrightarrow A(\alpha)).$$

Proposition 5. (GV) Let $\mathbf{M}_2^-$ be the theory resulting from $\mathbf{M}_2$ by replacing $\mathrm{AC}_{10}!$ by qf-AC_{10} (or equivalently by qf-$\mathrm{AC}_{10}!$). Then

(a) $\mathbf{M}_2 = \mathbf{M}_2^- + \mathrm{CF}_1$.

(b) $\mathbf{HA}_2$ + qf-AC_{10} + CF_1 entails $\mathrm{AC}_{10}!$, where $\mathbf{HA}_2$ has symbols and axioms for all primitive recursive functions of type 2, with extensional equality.

(c) CF_1 is not provable in $\mathbf{M}_2^-$ or in $\mathbf{HA}_2$ + qf-AC_{10}.

In the two-sorted language, Kleene expressed continuous choice in terms of an intelligent modulus of continuity for a choice functional. "Weak continuity" and "continuous non-choice" partially separated the roles of continuity and choice.
In the three-sorted language, continuous choice can be naturally decomposed into a classically correct choice axiom schema

AC$_{10}$: $\forall\alpha\exists y\, A(\alpha, y) \rightarrow \exists\Psi\forall\alpha A(\alpha, \Psi(\alpha))$

and an intuitionistic continuity principle $\forall\Phi \mathrm{Cont}(\Phi)$.

In turn, AC_{10} can be decomposed into a *bounded choice schema*

BC$_{10}$: $\forall\alpha\exists y \leq \Phi(\alpha)\, A(\alpha, y) \rightarrow \exists\Psi\forall\alpha A(\alpha, \Psi(\alpha))$

and a *bounding axiom schema*

AB$_{10}$: $\forall\alpha\exists y A(\alpha, y) \rightarrow \exists\Phi\forall\alpha\exists y \leq \Phi(\alpha)\, A(\alpha, y)$.

AB_{10} guarantees that every continuous functional has a modulus of continuity, but the modulus is not required to code the functional.

Challenge: Does $\mathbf{M}_2 + \mathrm{BC}_{10}$ prove AB_{10}?

Kleene specified some *primitive recursive coding*:

- $(y_0, \ldots, y_n) = 2^{y_0} \cdot \ldots \cdot p_n^{y_n}$ where p_n is the nth prime.
- $(y)_n$ is the exponent of p_n in the prime factorization of y.
- $lh(y) = \Sigma_{n<y} sg((y)_n)$ (the number of nonzero exponents in the prime factorization of y).
- $Seq(y) \equiv \forall n < lh(y)\,(y)_n > 0$.
- $\langle\,\rangle = 1$ and $\langle x_0, \ldots, x_n \rangle = (x_0 + 1, \ldots, x_n + 1)$.
- $*$ denotes concatenation of sequence numbers.
- $\overline{\alpha}(0) = \langle\,\rangle$ and $\overline{\alpha}(y+1) = \langle \alpha(0), \ldots, \alpha(y) \rangle$.

In the three-sorted language, countable choice can be stated

$\mathbf{AC}_{02}$: $\qquad \forall x \exists \Phi A(x, \Phi) \rightarrow \exists \Psi \forall x A(x, (\lambda \beta . \Psi(\lambda t . x, \beta)))$

where $\Psi(\alpha, \beta) \equiv \Psi(\lambda t.(\alpha(t), \beta(t))) \equiv \Psi(\lambda t.(2^{\alpha(t)} \cdot 3^{\beta(t)}))$.

Challenge: Show that $\mathbf{M}_2$ does not prove $\mathbf{AC}_{02}$.

As Kohlenbach [2002] observed, continuity properties of functionals can be directly expressed in the three-sorted language, for example

- "Φ is (pointwise) continuous at α" by $\mathrm{Cont}_\alpha(\Phi)$: $\exists y \forall \beta [\overline{\beta}(y) = \overline{\alpha}(y) \rightarrow \Phi(\beta) = \Phi(\alpha)]$.
- "Φ is sequentially continuous at α" by $\mathrm{SeqCont}_\alpha(\Phi)$: $\forall \beta [\forall n \overline{(\beta)_n}(n) = \overline{\alpha}(n) \rightarrow \exists n \forall m > n\, \Phi((\beta)_m) = \Phi(\alpha)]$.
- "Φ is effectively discontinuous at α" by $\mathrm{EffDiscont}_\alpha(\Phi)$: $\forall n \exists \beta [\overline{\beta}(n) = \overline{\alpha}(n) \;\&\; \Phi(\beta) \neq \Phi(\alpha)]$.
- "Φ is continuous" by $\mathrm{Cont}(\Phi) \equiv \forall \alpha \mathrm{Cont}_\alpha(\Phi)$, etc.

Proposition 7. (a) $\mathbf{IA}_2$ proves $\forall \alpha [\mathrm{Cont}_\alpha(\Phi) \rightarrow \mathrm{SeqCont}_\alpha(\Phi)]$.

(b) $\mathbf{IA}_2$ + qf-AC_{00} proves $\mathrm{SeqCont}(\Phi) \rightarrow \forall \beta \neg \mathrm{EffDiscont}_\beta(\Phi)$.

(c) $\mathbf{IA}_2$ + qf-AC_{00} + qf-(*) proves $\mathrm{SeqCont}(\Phi) \rightarrow \neg\neg \mathrm{Cont}(\Phi)$, where (*) is $\forall x \neg\neg \exists y A(x, y) \rightarrow \neg\neg \forall x \exists y A(x, y)$.

qf-(*) is weaker than Markov's Principle, as $\mathbf{FIM}$ + qf-(*) $\nvdash$ $\mathrm{MP}^\vee$.

Kohlenbach proves $\mathrm{SeqCont}(\Phi) \to \mathrm{Cont}(\Phi)$ in a weak classical three-sorted theory ($\mathrm{RCA}_0^2 = \text{E-PRA}^2 + \text{qf-AC}_{00}$) with restricted induction.

We were unable to prove this in $\mathbf{IA}_2 + \text{qf-AC}_{00} + \mathrm{MP}$. However,

Proposition 8. $\mathbf{IA}_2 + \text{qf-AC}_{00} + \Pi^0_1\text{-MP}$ proves

$$\mathrm{SeqCont}(\Phi) \to \mathrm{Cont}(\Phi),$$

where Π^0_1-**MP** is like Markov's Principle for Π^0_1 relations:

$$\neg\neg\exists x \forall y \alpha(x, y) = 0 \to \exists x \forall y \alpha(x, y) = 0.$$

Conjecture 9. $\mathbf{IA}_2 + \text{qf-AC}_{00} + \forall\Phi[\mathrm{SeqCont}(\Phi) \to \mathrm{Cont}(\Phi)]$ does not prove Π^0_1-MP.

Argument. The natural three-sorted intuitionistic system $\mathbf{FIM}_2$ should prove $\mathbf{IA}_2 + \text{qf-AC}_{00} + \forall\Phi\mathrm{Cont}(\Phi)$ but not Π^0_1-MP.

Kohlenbach uses Grilliot's trick to prove that over RCA_0^2 the axiom

($\exists^2$): $$\exists\Theta\forall\alpha(\Theta(\alpha) = 0 \leftrightarrow \exists x\alpha(x) = 0)$$

is equivalent to $\exists\Phi\neg\mathsf{SeqCont}(\Phi)$ and entails comprehension for all three-sorted formulas with only arithmetical quantifiers.

Over $\mathbf{M}_2$ (with intuitionistic logic) ($\exists^2$) is equivalent to full Σ_1^0-LEM, which entails the Law of Excluded Middle for all formulas of the three-sorted language having only arithmetical quantifiers, in particular $\forall\alpha[\exists x\alpha(x) \neq 0 \vee \forall x\alpha(x) = 0]$, which conflicts with INT and RUSS. Kohlenbach's article deserves a complete analysis from the constructive viewpoint.

Recent precise work by Ishihara, Josef Berger, Iris Loeb and Hannes Diener, and of course Veldman's reverse intuitionistic mathematical studies of Brouwer's bar and fan theorems, can readily be incorporated into our general framework. At the very least, Vafeiadou's precise comparison of minimal constructive formalisms helps to clarify a rapidly developing subject.

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