NOTE ON $\Pi^0_{n+1}$-LEM, $\Sigma^0_{n+1}$-LEM AND $\Sigma^0_{n+1}$-DNE

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Abstract. In [1] Akama, Berardi, Hayashi and Kohlenbach used a monotone modified realizability interpretation to establish the relative independence of $\Sigma^0_{n+1}$-DNE from $\Pi^0_{n+1}$-LEM over \textbf{HA}, and hence the independence of $\Sigma^0_{n+1}$-LEM from $\Pi^0_{n+1}$-LEM over \textbf{HA}, for all $n \geq 0$. We show that the same relative independence results hold for these arithmetical principles over Kleene and Vesley’s system \textbf{FIM} of intuitionistic analysis [3], which extends \textbf{HA} and is consistent with \textbf{PA} but not with classical analysis. The double negations of the closures of $\Sigma^0_{n+1}$-LEM, $\Sigma^0_{n+1}$-DNE and $\Pi^0_{n+1}$-LEM are also considered, and shown to behave differently with respect to \textbf{HA} and \textbf{FIM}. Various elementary questions remain to be answered.

Definitions of the Arithmetical Principles. Unless otherwise noted, “LEM” (Law of Excluded Middle), “DNE” (Double Negation Elimination), and “LLPO” (Lesser Limited Principle of Omniscience) denote the (universal closures of the) purely arithmetical schemas, without function variables. If $\Phi$ is $\Sigma^0_n$ or $\Pi^0_n$ for some $n \geq 1$ then

(i) $\Phi$-LEM is $A \lor \neg A$ where $A \in \Phi$.
(ii) $\Phi$-DNE is $\neg \neg A \rightarrow A$ where $A \in \Phi$.
(iii) $\Phi$-LLPO is $\neg (A \land B) \rightarrow (C \lor D)$, where $A, B \in \Phi$ and $C, D$ are the duals of $A, B$ respectively.
(iv) $\Delta^0_n$-LEM is $(A \leftrightarrow B) \rightarrow (B \lor \neg B)$ where $A \in \Pi^0_n$ and $B \in \Sigma^0_n$.

The precise statement of $\Delta^0_n$-LEM is important, since $\Sigma^0_{n+1}$-DNE is equivalent over $\textbf{HA} + \Sigma^0_n$-LEM to the schema $(\neg A \leftrightarrow B) \rightarrow (A \lor \neg A)$ where $A, B \in \Sigma^0_n$. Kleene used this principle for $n = 0$ to prove that every $\Delta^0_1$ relation is recursive. The corresponding observation for $n \geq 0$ is the Kleene-Post-Mostowski Theorem.

1. Some Results of Akama, Berardi, Hayashi and Kohlenbach Extended to \textbf{FIM}

Lemma 1. The following are equivalent, for any theory $T \supseteq \textbf{HA}$:

(i) $T + \Pi^0_1$-LEM proves $\Sigma^0_1$-LEM.
(ii) $T + \Pi^0_1$-LEM proves Markov’s Principle $\Sigma^0_1$-DNE.

Proof. (i) $\Rightarrow$ (ii) holds because decidable predicates are stable under double negation. (ii) $\Rightarrow$ (i) holds because

$\forall x \neg R(x) \lor \neg \forall x \neg R(x) \land [\neg \exists x R(x) \rightarrow \exists x R(x)] \rightarrow [\exists x R(x) \lor \neg \exists x R(x)]$.

Now let $T(e, x, y)$ be a quantifier-free formula numeralwise expressing in $\textbf{HA}$ (hence also in \textbf{FIM}) the Kleene T-predicate, and let $z \leq U(y)$ be a quantifier-free formula numeralwise expressing in $\textbf{HA}$ (hence also in \textbf{FIM}) the relation “$z \leq U(y)$” where $U(y)$ is the value computed by the computation with gödel number $y$, or the gödel number of $y$ if $y$ is not the gödel number of a computation. With Kleene’s coding $\textbf{HA}$ proves $\forall e \forall x \forall y[T(e, x, y) \rightarrow \forall z(z \leq U(y) \rightarrow \neg T(e, x, z))]$, and we will use this property to prove the next lemma.

I am grateful to Ulrich Kohlenbach for pointing me to [1], and to the organizers of the 2005 Oberwolfach conference on Proof Theory and Constructive Mathematics for a terrific mathematical experience.
Lemma 2. HA (hence also FIM) proves
\[ \forall f \neg \forall x \exists y [T(f, x, y) \wedge [\forall z \leq U(y) \neg T(x, x, z) \rightarrow \forall y \neg T(x, y)]]. \]

Proof. Assume for contradiction
\[ \forall x \exists y [T(f, x, y) \wedge [\forall z \leq U(y) \neg T(x, x, z) \rightarrow \forall y \neg T(x, y)]]. \]

After \(\forall\)-elimination assume for \(\exists y\)-elimination:
\[ T(f, f, y) \wedge [\forall z \leq U(y) \neg T(f, f, z) \rightarrow \forall y \neg T(f, f, y)], \]
from which \(T(f, f, y) \wedge \forall y \neg T(f, f, y)\) follows by the remark on coding.

FIM satisfies the “independence of (stable) premise” rule IPR:

\[ (\ast) \text{ If } \vdash_{\text{FIM}} (\neg A \rightarrow \exists x B(x)) \text{ then } \vdash_{\text{FIM}} \exists x (\neg A \rightarrow B(x)) \]

where \(x\) is not free in \(A\). The beautiful proof by Visser that HA is closed under IPR (cf. p. 138 of [6]) works also for FIM. If one uses the monotone form (“27.13 in [3]) of the bar induction schema, it is straightforward to show that FIM proves the Friedman translation of each of its mathematical axioms, and the logical rules and axioms behave as usual.

Lemma 3. FIM + \(\Pi^0_1\)-LEM does not prove \(\Sigma^0_1\)-LEM.

Proof. We use without much comment the fact that quantifier-free formulas are decidable and stable in FIM. Since primitive recursive codes for finite sequences of natural numbers are available in HA and hence in FIM, to prove the lemma we need only derive a contradiction from the assumption that \(\forall x[\forall y \neg R(x, y) \lor \exists y R(x, y)]\) is derivable in FIM from the universal closures of finitely many instances \(\forall x P_i(x, z) \lor \neg \forall x P_i(x, z), 1 \leq i \leq k\), of \(\Pi^0_1\)-LEM, where \(R(x, y)\) is \(T(x, x, y)\) and the \(P_i(x, z)\) are quantifier-free. Assume such a derivation exists, and let \(D(z)\) abbreviate \(\bigwedge_{i=1}^k (\forall x P_i(x, z) \lor \neg \forall x P_i(x, z))\). Then by the deduction theorem, FIM proves

\[ (i) \ \forall z D(z) \rightarrow \forall x [\forall y \neg R(x, y) \lor \exists y R(x, y)]. \]

We can construct a purely arithmetical formula \(E(w, z)\), with no \(\exists\) and no \(\forall\), such that FIM proves

\[ (ii) E(w, z) \iff \neg \neg E(w, z) \text{ and } \]

\[ (iii) E(\sigma(k), z) \iff \]

\[ \left[ \bigwedge_{i=1}^k \{ (\forall x P_i(x, z) : \sigma(i-1) > 0 \} \cup \{ \neg \forall x P_i(x, z) : \sigma(i-1) = 0 \} \right] \]

whence

\[ (iv) \ \forall z [D(z) \iff \exists \sigma \in \omega^2 E(\sigma(k), z)] \]

and so

\[ (v) \ \forall z \exists \sigma \in \omega^2 E(\sigma(k), z) \rightarrow \forall x [\forall y \neg R(x, y) \lor \exists y R(x, y)]. \]

The countable axiom of choice, which is an axiom schema of FIM, gives

\[ (vi) \forall \exists \sigma \in \omega^2 E(\sigma(k), z) \iff \exists \forall z (\forall t. \tau((z, t)) \in \omega^2 \wedge E(\lambda t. \tau((z, t)))(k), z)) \]

and hence

\[ (vii) \ \forall \forall \in \omega^2 [\forall \exists E(\lambda t. \tau((z, t)))(k), z) \rightarrow \forall x [\forall y \neg R(x, y) \lor \exists y R(x, y)] \]

where neither \(x\) nor \(y\) is free in the hypothesis, so also

\[ (viii) \ \forall x \forall \forall \in \omega^2 [\forall \exists E(\lambda t. \tau((z, t)))(k), z) \rightarrow \exists y [\forall y \neg R(x, y) \lor R(x, y)] \]

with a stable hypothesis. Applying \((\ast)\), FIM proves

\[ (ix) \ \forall \forall \forall \in \omega^2 \exists y [\forall \exists E(\lambda t. \tau((z, t)))(k), z) \rightarrow [\forall y \neg R(x, y) \lor R(x, y)]. \]
The classically false form of Brouwer’s Fan Theorem ("27.7 in [3]"), followed by the obvious counting argument, allows us to conclude from (ix) that FIM proves

\[(x) \forall x \exists m \forall \tau \in \omega \exists z E(\langle x, \tau \rangle, z) \rightarrow \exists y \leq m [\forall y \neg R(x, y) \lor R(x, y)]\]

and hence

\[(xi) \forall x \exists m [\forall z \exists \sigma \in \omega 2 E(\sigma, z) \rightarrow \exists y \leq m [\forall y \neg R(x, y) \lor R(x, y)]]\]

or equivalently

\[(xii) \forall x \exists m [\forall z D(z) \rightarrow \exists y \leq m [\forall y \neg R(x, y) \lor R(x, y)]]\]

But then by Kleene’s Rule FIM proves

\[(xiii) \forall x \exists y (T(f, x, y) \land (\forall z D(z) \rightarrow \exists z \leq U(y) [\forall y \neg T(x, y, z)]))\]

for some natural number \(f\), and hence

\[(xiv) \forall z D(z) \rightarrow \exists f F(f)\]

where \(F(f) = \forall x \exists y (T(f, x, y) \land [\forall z \leq U(y) \neg T(x, y, z) \lor \forall y \neg T(x, y, z)])\). Lemma 2 and (xiv) together now imply that FIM proves \(\forall z D(z)\), which is impossible since PA is consistent with FIM.

**Theorem 1.** (a) Each of the arithmetical principles \(\Sigma^0_n\)-LEM, \(\Sigma^0_n\)-DNE is independent relative to the arithmetical principle \(\Pi^0_n\)-LEM over FIM.

(b) For every \(n \geq 1\): Each of the arithmetical principles \(\Sigma^0_{n+1}\)-LEM, \(\Sigma^0_{n+1}\)-DNE is independent relative to the arithmetical principle \(\Pi^0_{n+1}\)-LEM over FIM + \(\Sigma^0_n\)-LEM.

**Proof.** (a) follows from Lemmas 1-3. To prove (b) for \(n \geq 1\), we need to generalize the lemmas. Since \(\Pi^0_{n+1}\)-LEM implies \(\Sigma^0_n\)-DNE and \(\Sigma^0_n\)-LEM, Lemma 1 holds with \(\Pi^0_{n+1}\) and \(\Sigma^0_{n+1}\) in place of \(\Pi^0_1\) and \(\Sigma^0_1\) respectively. Lemma 2 holds with \(T^Q\) in place of \(T\), where \(Q\) is any \(\Sigma^0_n\) predicate.

For Lemma 3 with FIM + \(\Sigma^0_n\)-LEM in place of FIM, and \(\Pi^0_{n+1}\) and \(\Sigma^0_{n+1}\) in place of \(\Pi^0_1\) and \(\Sigma^0_1\), take \(R(x, y)\) to be the complete predicate for arithmetical \(\Pi^0_n\). Each \(P_i(x, z)\) (now \(\Sigma_1^0\)) is equivalent in \(HA + \Sigma^0_n\)-LEM to its Gödel-Gentzen negative translation, so we may use these in defining \(E(w, z)\). FIM + \(\Sigma^0_n\)-LEM satisfies (*) because \(\Sigma^0_n\)-LEM proves its own Friedman translation by a stable formula. The step corresponding to (xii) \(\Rightarrow\) (xiii) is justified by Theorem 50(b) and Corollary 57 in [2], and the contradiction follows because PA is consistent with FIM + \(\Sigma^0_n\)-LEM.

**Corollary.** All the derivability and relative independence results over HA established by Akama, Berardi, Hayashi and Kohlenbach among the purely arithmetical principles \(\Delta^0_{n+1}\)-LEM, \(\Pi^0_{n+1}\)-LEM, \(\Sigma^0_{n+1}\)-DNE and \(\Sigma^0_{n+1}\)-DNE hold also over FIM, for every \(n \geq 0\).

**Proof.** The relative derivability results are preserved because HA is a subsystem of FIM. \(\Sigma^0_{n+1}\)-LLPO is independent relative to \(\Sigma^0_{n+1}\)-DNE over FIM because every theorem of FIM + \(\Sigma^0_{n+1}\)-DNE is classically realizable by a \(\Delta^0_n\) function, while \(\Sigma^0_{n+1}\)-LLPO is not. Hence also \(\Pi^0_{n+1}\)-LEM and \(\Sigma^0_{n+1}\)-LEM are independent relative to \(\Sigma^0_{n+1}\)-DNE over FIM.

The theorem takes care of the other cases. For example, \(\Sigma^0_{n+1}\)-DNE is independent relative to \(\Delta^0_{n+1}\)-LEM over FIM by the theorem, because FIM + \(\Pi^0_{n+1}\)-LEM proves \(\Delta^0_{n+1}\)-LEM but not \(\Sigma^0_{n+1}\)-DNE.

**Open Questions?** I do not know whether \(\Pi^0_{n+1}\)-LEM is independent relative to \(\Sigma^0_{n+1}\)-LLPO over FIM. Lifschitz realizability cannot be used here because FIM includes countable and continuous choice principles. I also do not know whether \(\Delta^0_{n+1}\)-LEM is independent of \(\Sigma^0_{n+1}\)-LEM over FIM. Classically, \(\Delta^0_{n+1}\)-LEM is realizable, \(\varepsilon_0\)realizable and \(\varepsilon_\omega\)realizable so these standard methods do not give independence even for \(n = 0\).
2. HOW DOUBLE NEGATION CHANGES THE PICTURE

Let \( \neg \neg \forall (\Sigma_n^0 \text{-LEM}) \) abbreviate the double negation of the universal closure of arithmetical \( \Sigma_n^0 \text{-LEM} \), and similarly for the other principles. For each \( n \geq 0 \) the weaker principles behave, with respect to relative independence over \( \text{HA} \), very much like the stronger ones.

**Theorem 2.** Over \( \text{HA} \), for each \( n \geq 1 \):

(a) \( \neg \neg \forall (\Sigma_n^0 \text{-LEM}) \) entails \( \neg \neg \forall (\Pi_n^0 \text{-LEM}) \).

(b) \( \neg \neg \forall (\Pi_n^0 \text{-LEM}) \) entails \( \neg \neg \forall (\Delta_n^0 \text{-LEM}) \), but not conversely.

(c) \( \neg \neg \forall (\Sigma_n^0 \text{-LEM}) \) entails \( \neg \neg \forall (\Sigma_n^0 \text{-DNE}) \), but not conversely.

(d) \( \neg \neg \forall (\Sigma_n^0 \text{-DNE}) \) entails \( \neg \neg \forall (\Delta_n^0 \text{-LEM}) \), but not conversely.

(e) \( \neg \neg \forall (\Sigma_n^0 \text{-DNE}) \) does not entail \( \neg \neg \forall (\Pi_n^0 \text{-LEM}) \).

**Proof.** Only the relative independence results require comment. Classical number-realizability relativized to \( \Delta_n^0 \) shows that \( \text{HA} + \Delta_n^0 \text{-LEM} \) does not prove \( \neg \neg \forall (\Pi_n^0 \text{-LEM}) \), and that \( \text{HA} + \Sigma_n^0 \text{-DNE} \) proves neither \( \neg \neg \forall (\Sigma_n^0 \text{-LEM}) \) nor \( \neg \neg \forall (\Pi_n^0 \text{-LEM}) \). To show \( \text{HA} + \Delta_n^0 \text{-LEM} \) does not prove \( \neg \neg \forall (\Sigma_n^0 \text{-DNE}) \) use modified number-realizability relativized to \( \Delta_n^0 \).

Does \( \text{HA} + \Pi_n^0 \text{-LEM} \) or \( \text{FIM} + \Pi_n^0 \text{-LEM} \) prove either \( \neg \neg \forall (\Sigma_n^0 \text{-DNE}) \) or \( \neg \neg \forall (\Sigma_n^0 \text{-LEM}) \)? I do not know.

Most of Theorem 2 extends to \( \text{FIM} \), using \( \Delta_n^0 \)-realizability (a generalization of the \( G \)-realizability in [4]) for the nonderivabilities in (b) and (d). However, \( \neg \neg \forall (\Sigma_n^0 \text{-DNE}) \) is interderivable with \( \neg \neg \forall (\Sigma_n^0 \text{-LEM}) \) over \( \text{FIM} \), by the following result.

**Theorem 3.** (a) Over \( \text{FIM} \), and hence over \( \text{HA} \), each original principle (possibly excepting \( \Sigma_1^0 \text{-DNE} \) and \( \Delta_1^0 \text{-LEM} \)) is strictly stronger than its doubly negated closure.

(b) \( \text{FIM} + \Sigma_n^0 \text{-DNE} \) proves \( \neg \forall (\Sigma_n^0 \text{-LEM}) \), for \( n \geq 1 \).

(c) \( \text{HA} + \Sigma_n^0 \text{-DNE} \) does not prove \( \neg \forall (\Sigma_n^0 \text{-LEM}) \).

**Proofs.** Each doubly negated closure is classically function-realizable, while \( \Sigma_1^0 \text{-DNE} \) and \( \Delta_1^0 \text{-LEM} \) are the only original principles with this property, so (a) holds.

By an argument essentially due to Solovay, \( \text{FIM} + \Sigma_n^0 \text{-DNE} \) proves \( \neg \forall (\Sigma_n^0 \text{-LEM}) \) for every \( n \geq 1 \). The proof in [5] using an analytical version of Markov’s Principle can be paraphrased to give the result for the arithmetical principles from arithmetical \( \Sigma_n^0 \text{-DNE} \), so (b) holds also. Finally, (c) follows from the proof of Theorem 2(c).

**REFERENCES**


