NOTE ON Π^0_{n+1}-LEM, Σ^0_{n+1}-LEM AND Σ^0_{n+1}-DNE

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ABSTRACT. In [1] Akama, Berardi, Hayashi and Kohlenbach used a monotone modified realizability interpretation to establish the relative independence of Σ^0_{n+1}-DNE from Π^0_{n+1}-LEM over **HA**, and hence the independence of Σ^0_{n+1}-LEM from Π^0_{n+1}-LEM over **HA**, for all $n \geq 0$. We show that the same relative independence results hold for these arithmetical principles over Kleene and Vesley's system **FIM** of intuitionistic analysis [3], which extends **HA** and is consistent with **PA** but not with classical analysis.[1] The double negations of the closures of Σ^0_{n+1}-LEM, Σ^0_{n+1}-DNE and Π^0_{n+1}-LEM are also considered, and shown to behave differently with respect to **HA** and **FIM**. Various elementary questions remain to be answered.

Definitions of the Arithmetical Principles. Unless otherwise noted, "LEM" (Law of Excluded Middle), "DNE" (Double Negation Elimination), and "LLPO" (Lesser Limited Principle of Omniscience) denote the (universal closures of the) purely arithmetical schemas, without function variables. If Φ is Σ^0_n or Π^0_n for some $n \geq 1$ then

(i) Φ-LEM is $A \vee \neg A$ where $A \in \Phi$.

(ii) Φ-DNE is $\neg\neg A \to A$ where $A \in \Phi$.

(iii) Φ-LLPO is $\neg(A \wedge B) \to (C \vee D)$, where $A, B \in \Phi$ and C, D are the duals of A, B respectively.

(iv) Δ^0_n-LEM is $(A \leftrightarrow B) \to (B \vee \neg B)$ where $A \in \Pi^0_n$ and $B \in \Sigma^0_n$.

The precise statement of Δ^0_n-LEM is important, since Σ^0_{n+1}-DNE is equivalent over **HA** + Σ^0_n-LEM to the schema $(\neg A \leftrightarrow B) \to (A \vee \neg A)$ where $A, B \in \Sigma^0_{n+1}$. Kleene used this principle for $n = 0$ to prove that every Δ^0_1 relation is recursive. The corresponding observation for $n \geq 0$ is the Kleene-Post-Mostowski Theorem.

1. SOME RESULTS OF AKAMA, BERARDI, HAYASHI AND KOHLENBACH EXTENDED TO FIM

Lemma 1. The following are equivalent, for any theory $\mathbf{T} \supseteq \mathbf{HA}$:

(i) $\mathbf{T}$ + Π^0_1-LEM proves Σ^0_1-LEM.

(ii) $\mathbf{T}$ + Π^0_1-LEM proves Markov's Principle Σ^0_1-DNE.

Proof. $(i) \Rightarrow (ii)$ holds because decidable predicates are stable under double negation. $(ii) \Rightarrow (i)$ holds because

$$[\forall x \neg R(x) \vee \neg \forall x \neg R(x)] \ \& \ [\neg\neg\exists x R(x) \to \exists x R(x)] \to [\exists x R(x) \vee \neg \exists x R(x)] \ .$$

Now let $T(e, x, y)$ be a quantifier-free formula numeralwise expressing in **HA** (hence also in **FIM**) the Kleene T-predicate, and let $z \leq U(y)$ be a quantifier-free formula numeralwise expressing in **HA** (hence also in **FIM**) the relation "z ≤ U(y)" where U(y) is the value computed by the computation with gödel number y, or the gödel number of y if y is not the gödel number of a computation. With Kleene's coding **HA** proves $\forall e \forall x \forall y[T(e, x, y) \to \forall z(z \leq U(y) \to \neg T(e, x, z))]$, and we will use this property to prove the next lemma.

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Lemma 2. **HA** (hence also **FIM**) proves

$$\forall f \neg \forall x \exists y [T(f,x,y) \wedge [\forall z_{z \leq U(y)} \neg T(x,x,z) \rightarrow \forall y \neg T(x,x,y)]] \ .$$

Proof. Assume for contradiction

$$\forall x \exists y [T(f,x,y) \wedge [\forall z_{z \leq U(y)} \neg T(x,x,z) \rightarrow \forall y \neg T(x,x,y)]] \ .$$

After $\forall$-elimination assume for $\exists y$-elimination:

$$T(f,f,y) \wedge [\forall z_{z \leq U(y)} \neg T(f,f,z) \rightarrow \forall y \neg T(f,f,y)] \ ,$$

from which $T(f,f,y) \wedge \forall y \neg T(f,f,y)$ follows by the remark on coding.

FIM satisfies the "independence of (stable) premise" rule IPR:

$$(*) \text{ If } \vdash_{\mathbf{FIM}} (\neg A \rightarrow \exists x B(x)) \text{ then } \vdash_{\mathbf{FIM}} \exists x (\neg A \rightarrow B(x))$$

where x is not free in A. The beautiful proof by Visser that **HA** is closed under IPR (cf. p. 138 of [6]) works also for **FIM**. If one uses the monotone form (*27.13 in [3]) of the bar induction schema, it is straightforward to show that **FIM** proves the Friedman translation of each of its mathematical axioms, and the logical rules and axioms behave as usual.

Lemma 3. **FIM** + Π^0_1-LEM does not prove Σ^0_1-LEM.

Proof. We use without much comment the fact that quantifier-free formulas are decidable and stable in **FIM**. Since primitive recursive codes for finite sequences of natural numbers are available in **HA** and hence in **FIM**, to prove the lemma we need only derive a contradiction from the assumption that $\forall x [\forall y \neg R(x,y) \vee \exists y R(x,y)]$ is derivable in **FIM** from the universal closures of finitely many instances $\forall x P_i(x,z) \vee \neg \forall x P_i(x,z)$, $1 \leq i \leq k$, of Π^0_1-LEM, where $R(x,y)$ is $T(x,x,y)$ and the $P_i(x,z)$ are quantifier-free. Assume such a derivation exists, and let $D(z)$ abbreviate $\bigwedge_{i=1}^k (\forall x P_i(x,z) \vee \neg \forall x P_i(x,z))$. Then by the deduction theorem, **FIM** proves

$$\text{(i) } \forall z D(z) \rightarrow \forall x [\forall y \neg R(x,y) \vee \exists y R(x,y)] \ .$$

We can construct a purely arithmetical formula $E(w,z)$, with no $\exists$ and no $\vee$, such that **FIM** proves

(ii) $E(w,z) \leftrightarrow \neg\neg E(w,z)$ and
(iii) $E(\overline{\sigma}(\mathbf{k}),z) \leftrightarrow$
$\left[\bigwedge_{i=1}^k (\{\forall x P_i(x,z) : \sigma(i \dot{-} 1) > 0\} \cup \{\neg \forall x P_i(x,z) : \sigma(i \dot{-} 1) = 0\})\right]$

whence

$$\text{(iv) } \forall z \, [D(z) \leftrightarrow \exists \sigma \in {}^{\omega}2 \ E(\overline{\sigma}(\mathbf{k}),z)]$$

and so

$$\text{(v) } \forall z \exists \sigma \in {}^{\omega}2 \ E(\overline{\sigma}(\mathbf{k}),z) \rightarrow \forall x [\forall y \neg R(x,y) \vee \exists y R(x,y)] \ .$$

The countable axiom of choice, which is an axiom schema of **FIM**, gives

$$\text{(vi)} \forall z \exists \sigma \in {}^{\omega}2 \ E(\overline{\sigma}(\mathbf{k}),z) \leftrightarrow \exists \tau \forall z (\lambda t.\tau((z,t)) \in {}^{\omega}2 \wedge E(\overline{\lambda t.\tau((z,t))}(\mathbf{k}),z))$$

and hence

$$\text{(vii) } \forall \tau \in {}^{\omega}2 [\forall z \ E(\overline{\lambda t.\tau((z,t))}(\mathbf{k}),z) \rightarrow \forall x [\forall y \neg R(x,y) \vee \exists y R(x,y)]]$$

where neither x nor y is free in the hypothesis, so also

$$\text{(viii) } \forall x \forall \tau \in {}^{\omega}2 [\forall z \ E(\overline{\lambda t.\tau((z,t))}(\mathbf{k}),z) \rightarrow \exists y [\forall y \neg R(x,y) \vee R(x,y)]]$$

with a stable hypothesis. Applying $(*)$, **FIM** proves

$$\text{(ix) } \forall x \forall \tau \in {}^{\omega}2 \exists y [\forall z \ E(\overline{\lambda t.\tau((z,t))}(\mathbf{k}),z) \rightarrow [\forall y \neg R(x,y) \vee R(x,y)]] \ .$$

The classically false form of Brouwer's Fan Theorem (*27.7 in [3]), followed by the obvious counting argument, allows us to conclude from (ix) that **FIM** proves

$$\text{(x) } \forall x \exists m \forall \tau \in {}^{\omega}2 [\forall z\, E(\overline{\lambda t.\tau((z,t))}(\mathbf{k}), z) \to \exists y_{y \le m} [\forall y \neg R(x,y) \vee R(x,y)]]$$

and hence

$$\text{(xi) } \forall x \exists m [\forall z \exists \sigma \in {}^{\omega}2\; E(\overline{\sigma}(\mathbf{k}), z) \to \exists y_{y \le m} [\forall y \neg R(x,y) \vee R(x,y)]]$$

or equivalently

$$\text{(xii) } \forall x \exists m\, [\forall z D(z) \to \exists y_{y \le m} [\forall y \neg R(x,y) \vee R(x,y)]] \;\; .$$

But then by Kleene's Rule **FIM** proves

$$\text{(xiii) } \forall x \exists y \left(T(\mathbf{f}, x, y) \wedge \left(\forall z D(z) \to \exists z_{z \le U(y)} [\forall y \neg T(x,x,y) \vee T(x,x,z)]\right)\right)$$

for some natural number f, and hence

$$\text{(xiv) } \forall z D(z) \to \exists f F(f)$$

where $F(f)$ is $\forall x \exists y \left(T(f,x,y) \wedge [\forall z_{z \le U(y)} \neg T(x,x,z) \to \forall y \neg T(x,x,y)]\right)$. Lemma 2 and (xiv) together now imply that **FIM** proves $\neg \forall z D(z)$, which is impossible since **PA** is consistent with **FIM**.

Theorem 1. (a) Each of the arithmetical principles Σ^0_1-LEM, Σ^0_1-DNE is independent relative to the arithmetical principle Π^0_1-LEM over **FIM**.

(b) For every $n \ge 1$: Each of the arithmetical principles Σ^0_{n+1}-LEM, Σ^0_{n+1}-DNE is independent relative to the arithmetical principle Π^0_{n+1}-LEM over **FIM** + Σ^0_n-LEM.

Proof. (a) follows from Lemmas 1-3. To prove (b) for $n \ge 1$, we need to generalize the lemmas. Since Π^0_{n+1}-LEM implies Σ^0_n-DNE and Σ^0_n-LEM, Lemma 1 holds with Π^0_{n+1} and Σ^0_{n+1} in place of Π^0_1 and Σ^0_1 respectively. Lemma 2 holds with T^Q in place of T, where Q is any Σ^0_n predicate.

For Lemma 3 with **FIM** + Σ^0_n-LEM in place of **FIM**, and Π^0_{n+1} and Σ^0_{n+1} in place of Π^0_1 and Σ^0_1, take $R(x,y)$ to be the complete predicate for arithmetical Π^0_n. Each $P_i(x,z)$ (now Σ^0_n) is equivalent in **HA** + Σ^0_n-LEM to its Gödel-Gentzen negative translation, so we may use these in defining $E(w,z)$. **FIM** + Σ^0_n-LEM satisfies $(*)$ because Σ^0_n-LEM proves its own Friedman translation by a stable formula. The step corresponding to (xii) $\Rightarrow$ (xiii) is justified by Theorem 50(b) and Corollary 57 in [2], and the contradiction follows because **PA** is consistent with **FIM** + Σ^0_n-LEM.

Corollary. All the derivability and relative independence results over **HA** established by Akama, Berardi, Hayashi and Kohlenbach among the purely arithmetical principles Δ^0_{n+1}-LEM, Π^0_{n+1}-LEM, Σ^0_{n+1}-DNE and Σ^0_{n+1}-LEM hold also over **FIM**, for every $n \ge 0$.

Proof. The relative derivability results are preserved because **HA** is a subsystem of **FIM**. Σ^0_{n+1}-LLPO is independent relative to Σ^0_{n+1}-DNE over **FIM** because every theorem of **FIM** + Σ^0_{n+1}-DNE is classically realizable by a Δ^0_n function, while Σ^0_{n+1}-LLPO is not. Hence also Π^0_{n+1}-LEM and Σ^0_{n+1}-LEM are independent relative to Σ^0_{n+1}-DNE over **FIM**.

The theorem takes care of the other cases. For example, Σ^0_{n+1}-DNE is independent relative to Δ^0_{n+1}-LEM over **FIM** by the theorem, because **FIM** + Π^0_{n+1}-LEM proves Δ^0_{n+1}-LEM but not Σ^0_{n+1}-DNE.

Open Questions? I do not know whether Π^0_{n+1}-LEM is independent relative to Σ^0_{n+1}-LLPO over **FIM**. Lifschitz realizability cannot be used here because **FIM** includes countable and continuous choice principles. I also do not know whether Δ^0_{n+1}-LEM is independent of Σ^0_n-LEM over **FIM**. Classically, Δ^0_1-LEM is realizable, ${}_{\mathsf{S}}$realizable and ${}^{\mathsf{G}}$realizable so these standard methods do not give independence even for $n = 0$.

2. How Double Negation Changes the Picture

Let $\neg\neg\forall(\Sigma^0_n\text{-LEM})$ abbreviate the double negation of the universal closure of arithmetical Σ^0_n-LEM, and similarly for the other principles. For each $n \geq 0$ the weaker principles behave, with respect to relative independence over **HA**, very much like the stronger ones.

Theorem 2. Over **HA**, for each $n \geq 1$:
(a) $\neg\neg\forall(\Sigma^0_n\text{-LEM})$ entails $\neg\neg\forall(\Pi^0_n\text{-LEM})$.
(b) $\neg\neg\forall(\Pi^0_n\text{-LEM})$ entails $\neg\neg\forall(\Delta^0_n\text{-LEM})$, but not conversely.
(c) $\neg\neg\forall(\Sigma^0_n\text{-LEM})$ entails $\neg\neg\forall(\Sigma^0_n\text{-DNE})$, but not conversely.
(d) $\neg\neg\forall(\Sigma^0_n\text{-DNE})$ entails $\neg\neg\forall(\Delta^0_n\text{-LEM})$, but not conversely.
(e) $\neg\neg\forall(\Sigma^0_n\text{-DNE})$ does not entail $\neg\neg\forall(\Pi^0_n\text{-LEM})$.

Proof. Only the relative independence results require comment. Classical number-realizability relativized to Δ^0_n shows that **HA** + Δ^0_n-LEM does not prove $\neg\neg\forall(\Pi^0_n\text{-LEM})$, and that **HA** + Σ^0_n-DNE proves neither $\neg\neg\forall(\Sigma^0_n\text{-LEM})$ nor $\neg\neg\forall(\Pi^0_n\text{-LEM})$. To show **HA** + Δ^0_1-LEM does not prove $\neg\neg\forall(\Sigma^0_n\text{-DNE})$ use modified number-realizability relativized to Δ^0_n.

Does **HA** + Π^0_n-LEM or **FIM** + Π^0_n-LEM prove either $\neg\neg\forall(\Sigma^0_n\text{-DNE})$ or $\neg\neg\forall(\Sigma^0_n\text{-LEM})$? I do not know.

Most of Theorem 2 extends to **FIM**, using ${}^{\Delta^0_n}$realizability (a generalization of the ${}^{\mathsf{G}}$realizability in [4]) for the nonderivabilities in (b) and (d). However, $\neg\neg\forall(\Sigma^0_n\text{-DNE})$ is interderivable with $\neg\neg\forall(\Sigma^0_n\text{-LEM})$ over **FIM**, by the following result.

Theorem 3. (a) Over **FIM**, and hence over **HA**, each original principle (possibly excepting Σ^0_1-DNE and Δ^0_1-LEM) is strictly stronger than its doubly negated closure.
(b) **FIM** + Σ^0_n-DNE proves $\neg\neg\forall(\Sigma^0_n\text{-LEM})$, for $n \geq 1$.
(c) **HA** + Σ^0_n-DNE does not prove $\neg\neg\forall(\Sigma^0_n\text{-LEM})$.

Proofs. Each doubly negated closure is classically function-realizable, while Σ^0_1-DNE and Δ^0_1-LEM are the only original principles with this property, so (a) holds.

By an argument essentially due to Solovay, **FIM** + Σ^0_n-DNE proves $\neg\neg\forall(\Sigma^0_n\text{-LEM})$ for every $n \geq 1$. The proof in [5] using an analytical version of Markov's Principle can be paraphrased to give the result for the arithmetical principles from arithmetical Σ^0_n-DNE, so (b) holds also. Finally, (c) follows from the proof of Theorem 2(c).

References

1. Akama, Y., Berardi, S., Hayashi, S. and Kohlenbach, U.: An arithmetical hierarchy of the law of excluded middle and related principles. LICS (2004) 192–201
2. Kleene, S. C.: Formalized recursive functionals and formalized realizability. Amer. Math. Soc. Memoirs **89** (1969)
3. Kleene, S. C. and Vesley, R. E.: The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions. North-Holland, Amsterdam (1965)
4. Moschovakis, J. R.: Can there be no nonrecursive functions?. Jour. Symb. Logic **36** (1971) 309–315
5. Moschovakis, J. R.: Classical and constructive hierarchies in extended intuitionistic analysis. Jour. Symb. Logic **68** (2003) 1015–1043
6. Troelstra, A. S. and van Dalen, D.: Constructivism in Mathematics: An Introduction, I and II. North-Holland, Amsterdam (1988)