MARKOV'S PRINCIPLE AND SUBSYSTEMS OF INTUITIONISTIC ANALYSIS

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ABSTRACT. Using a technique developed by Coquand and Hofmann [2] we verify that adding the analytical form MP_1: $\forall\alpha(\neg\neg\exists x\alpha(x) = 0 \to \exists x\alpha(x) = 0)$ of Markov's Principle does not increase the class of Π^0_2 formulas provable in Kleene and Vesley's formal system for intuitionistic analysis, or in subsystems obtained by omitting or restricting various axiom schemas in specified ways.

INTRODUCTION

In [5] Kleene proved that Markov's Principle MP_1 is neither provable nor refutable in his formal system **I** for intuitionistic analysis. By the Friedman-Dragalin translation, Markov's Rule is admissible for **I** and many subsystems.

We show that adding MP_1 as an axiom to **I** does not increases consistency strength, in the sense that no additional Π^0_2 formulas become provable. The method adapted from [2] works also for subsystems of **I**, with a few interesting exceptions.

1. LANGUAGE, LOGIC, AND BASIC MATHEMATICAL AXIOMS

1.1. **The two-sorted formal language and intuitionistic predicate logic.** Kleene and Vesley's language $\mathcal{L}_1$ for two-sorted intuitionistic number theory or "intuitionistic analysis" has variables a,b,c,...,x,y,z,..., intended to range over natural numbers; variables $\alpha, \beta, \gamma, \ldots$, intended to range over one-place number-theoretic functions (choice sequences); finitely many constants $0, ', +, \cdot, \mathrm{f}_4, \ldots, \mathrm{f}_\mathrm{p}$, each representing a primitive recursive function or functional, where f_i has k_i places for number arguments and l_i places for type-1 function arguments; parentheses indicating function application; and Church's λ.

The *terms* (of type 0) and *functors* (of type 1) are defined inductively as follows. The number variables and 0 are terms. The function variables and each f_i with $k_i = 1, l_i = 0$ are functors. If $\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}$ are terms and $\mathrm{u}_1, \ldots, \mathrm{u}_{l_i}$ are functors, then $\mathrm{f}_i(\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}, \mathrm{u}_1, \ldots, \mathrm{u}_{l_i})$ is a term. If x is a number variable and t is a term, then λx.t is a functor. And if u is a functor and t is a term, then (u)(t) is a term.

There is one relation symbol = for equality between terms; equality between functors u, v is defined extensionally by $\mathrm{u} = \mathrm{v} \equiv \forall x(\mathrm{u}(x) = \mathrm{v}(x))$ (where x is not free in u or v). The atomic formulas of $\mathcal{L}_1$ are the expressions s = t where s, t are terms. Composite formulas are defined inductively, using the connectives $\&, \vee, \to, \neg$, quantifiers $\forall, \exists$ of both sorts, and parentheses (often omitted under the usual conventions on scope). $\mathrm{A} \leftrightarrow \mathrm{B}$ is defined by $(\mathrm{A} \to \mathrm{B}) \,\&\, (\mathrm{B} \to \mathrm{A})$.

The logical axioms and rules are those of two-sorted intuitionistic predicate logic, as presented in [5] (building on [3]). If the intuitionistic axiom schema $\neg\mathrm{A} \to (\mathrm{A} \to \mathrm{B})$ were replaced by $\neg\neg\mathrm{A} \to \mathrm{A}$ (of which Markov's Principle MP_1 is a special case), two-sorted classical predicate logic would result.

1.2. **Two-sorted intuitionistic arithmetic $\mathbf{IA}_1$.** This is a conservative extension, in the language $\mathcal{L}_1$, of the first-order intuitionistic arithmetic $\mathbf{IA}_0$ in [3] based on $=, 0, ', +, \cdot$. The mathematical axioms of $\mathbf{IA}_1$ are:

(a) The axiom-schema of mathematical induction (for all formulas of $\mathcal{L}_1$): $\mathrm{A}(0) \ \& \ \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \to \mathrm{A}(\mathrm{x}')) \to \mathrm{A}(\mathrm{x})$.

(b) The axioms of $\mathbf{IA}_0$ for $=, 0, ', +, \cdot$ (axioms 14-21 on page 82 of [3]) and the axioms expressing the primitive recursive definitions of the additional function constants $\mathrm{f}_4, \ldots, \mathrm{f}_{26}$ given in [5] and [4].[1]

(c) The open equality axiom: $\mathrm{x} = \mathrm{y} \to \alpha(\mathrm{x}) = \alpha(\mathrm{y})$.

(d) The axiom-schema of λ-conversion: $(\lambda \mathrm{x}.\mathrm{t}(\mathrm{x}))(\mathrm{s}) = \mathrm{t}(\mathrm{s})$, where $\mathrm{t}(\mathrm{x})$ is a term and s is free for x in $\mathrm{t}(\mathrm{x})$.

For readers familiar with [5], $\mathbf{IA}_1$ is the subsystem of the "basic system" **B** obtained by omitting the axiom schemas of countable choice and bar induction (^{x}2.1 and ^{x}26.3, respectively).

$\mathbf{IA}_1$ can only prove the existence of primitive recursive sequences, in the sense that each closed theorem of the form $\exists \alpha \mathrm{A}(\alpha)$ has a primitive recursive witness. The finite list of primitive recursive function constants, with their corresponding axioms, is intended to be expanded as needed. Here we use the λ notation to explicitly define termwise multiplication of sequences. Let $(\alpha \cdot \beta)$ abbreviate $\lambda \mathrm{x}(\alpha(\mathrm{x}) \cdot \beta(\mathrm{x}))$.

1.3. **Intuitionistic recursive analysis IRA.** The principle of countable choice for numbers is expressed in $\mathcal{L}_1$ by the schema (*2.2 in [5]):

$$\mathrm{AC}_{00}: \quad \forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \to \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x})),$$

where α, x must be free for y in $\mathrm{A}(\mathrm{x}, \mathrm{y})$. Intuitionistic recursive analysis **IRA** can be axiomatized, as a subsystem of Kleene and Vesley's **B**, by $\mathbf{IA}_1$ + qf-AC_{00}, where qf-AC_{00} is the restriction of AC_{00} to formulas $\mathrm{A}(\mathrm{x}, \mathrm{y})$ without sequence quantifiers and with only bounded number quantifiers. **IRA** ensures that the range of the type-1 variables contains all general recursive sequences and is closed under general recursive processes. Troelstra's **EL** and Veldman's **BIM** are alternative axiomatizations of **IRA**, cf. [7], [6].

In the two-sorted language, **IRA** + MP_1 + CT_1 formalizes Russian recursive analysis (**RUSS** in [1]), where MP_1 is the functional form of Markov's Principle

$$\mathrm{MP}_1: \quad \forall \alpha[\neg\neg \exists \mathrm{x} \alpha(\mathrm{x}) = 0 \to \exists \mathrm{x} \alpha(\mathrm{x}) = 0]$$

and CT_1 expresses Church's Thesis:

$$\mathrm{CT}_1: \quad \forall \alpha \exists \mathrm{e} \forall \mathrm{x} \exists \mathrm{y}[\mathrm{T}_0(\mathrm{e}, \mathrm{x}, \mathrm{y}) \ \& \ \mathrm{U}(\mathrm{y}) = \alpha(\mathrm{x})].$$

[1] $\mathrm{f}_0 - \mathrm{f}_3$ are $0, ', +, \cdot$ respectively. $\mathrm{f}_4(\mathrm{a}, \mathrm{b}) = \mathrm{a}^{\mathrm{b}}$ (exponentiation), and $\mathrm{f}_5, \ldots, \mathrm{f}_{20}$ represent the primitive recursive function(al)s $a!$, $a \dot{-} b$, $pd(a)$, $min(a, b)$, $max(a, b)$, $\overline{sg}(a)$, $sg(a)$, $|a - b|$, $rm(a, b)$, $[a/b]$, $\Sigma_{y<b}\alpha(y)$, $\Pi_{y<b}\alpha(y)$, $min_{y \le b}\alpha(y)$, $max_{y \le b}\alpha(y)$, p_a (the a^{th} prime, with $p_0 = 2$), and $(a)_i$ (the exponent of p_i in the prime factorization of a) respectively. We write $(\mathrm{a})_{\mathrm{i}}$ for $\mathrm{f}_{20}(\mathrm{a}, \mathrm{i})$, and similarly for the other function constants. $\mathrm{f}_{21}(\mathrm{a}) = \mathrm{lh}(\mathrm{a})$ denotes the number $\Sigma_{i<a} sg((a)_i)$ of positive exponents in the prime factorization of a. Bounded quantifiers are defined with the help of bounded sum and product. Seq(a) is a prime formula equivalent to $\mathrm{a} > 0 \ \& \ \forall \mathrm{i} < \mathrm{lh}(\mathrm{a}) \ (\mathrm{a})_{\mathrm{i}} > 0$, expressing "$a$ codes the finite sequence $((a)_0 - 1, \ldots, (a)_{lh(a)-1} - 1)$". $\mathrm{f}_{??}(\mathrm{a}, \mathrm{b}) = \mathrm{a} * \mathrm{b}$ produces a code for the concatenation of two finite sequences from their codes: $\langle \mathrm{x}_0 + 1, \ldots, \mathrm{x}_\mathrm{k} + 1 \rangle * \langle \mathrm{x}_{\mathrm{k}+1} + 1, \ldots, \mathrm{x}_\mathrm{m} + 1 \rangle = \langle \mathrm{x}_0 + 1, \ldots, \mathrm{x}_\mathrm{m} + 1 \rangle$. (This coding is not onto but is one-to-one; in contrast, $\langle a_0, \ldots, a_k \rangle = \Pi_{i<k+1} p_i^{a_i} = \langle a_0, \ldots, a_k, 0 \rangle$.) $\langle \, \rangle = 1$ codes the empty sequence, and $\overline{\alpha}(\mathrm{x}) = \Pi_{\mathrm{i}<\mathrm{x}} \mathrm{p}_{\mathrm{i}}^{\alpha(\mathrm{i})+1}$ represents the standard code for the x^{th} initial segment of α.

The general recursive functions form a classical ω-model of **RUSS** and hence of **IRA**, but **RUSS** + AC_{00} (unlike **IRA** + AC_{00}) is inconsistent with classical logic.

2. Definition of the Translation, and Properties Proved in $\mathbf{IA}_1$

2.1. **Definition.** Let $\mathrm{Z}(\alpha)$ abbreviate $\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$. To each formula E of $\mathcal{L}_1$ and each sequence variable α not occurring in E, we associate another formula E^α with the same free variables plus α, by induction on the logical form of E as follows. For cases 4 and 5, β should be distinct from α. For case 4, $\mathrm{B}^{\alpha\cdot\beta}$ is the result of substituting $\alpha \cdot \beta$ for γ in the definition of B^γ, where γ does not occur in B.[2]

(1) P^α is $\mathrm{P} \vee \mathrm{Z}(\alpha)$ if P is prime.
(2) $(\mathrm{A} \ \&\ \mathrm{B})^\alpha$ is $\mathrm{A}^\alpha \ \&\ \mathrm{B}^\alpha$.
(3) $(\mathrm{A} \vee \mathrm{B})^\alpha$ is $\mathrm{A}^\alpha \vee \mathrm{B}^\alpha$.
(4) $(\mathrm{A} \to \mathrm{B})^\alpha$ is $\forall \beta \in 2^{\mathbb{N}}(\mathrm{A}^\beta \to \mathrm{B}^{\alpha\cdot\beta})$.
(5) $(\neg\mathrm{A})^\alpha$ is $\forall \beta \in 2^{\mathbb{N}}(\mathrm{A}^\beta \to \mathrm{Z}(\alpha\cdot\beta))$.
(6) $(\forall \mathrm{x} \mathrm{A}(\mathrm{x}))^\alpha$ is $\forall \mathrm{x} \mathrm{A}^\alpha(\mathrm{x})$.
(7) $(\exists \mathrm{x} \mathrm{A}(\mathrm{x}))^\alpha$ is $\exists \mathrm{x} \mathrm{A}^\alpha(\mathrm{x})$.
(8) $(\forall \gamma \mathrm{A}(\gamma))^\alpha$ is $\forall \gamma \mathrm{A}^\alpha(\gamma)$.
(9) $(\exists \gamma \mathrm{A}(\gamma))^\alpha$ is $\exists \gamma \mathrm{A}^\alpha(\gamma)$.

2.2. **Proposition.** $\mathbf{IA}_1 \vdash \forall \alpha, \beta \in 2^{\mathbb{N}}(\mathrm{Z}(\alpha\cdot\beta) \leftrightarrow \mathrm{Z}(\alpha) \vee \mathrm{Z}(\beta))$.

2.3. **Lemma.** $\mathbf{IA}_1 \vdash \forall \alpha, \beta, \gamma \in 2^{\mathbb{N}}(\mathrm{E}^\alpha \ \&\ \gamma = \alpha\cdot\beta \to \mathrm{E}^\gamma)$.

Proof. Only Cases 4 and 5 require attention. If E is $\mathrm{A} \to \mathrm{B}$ where A, B both satisfy the lemma, assume $\alpha, \beta, \gamma \in 2^{\mathbb{N}}$ & $(\mathrm{A} \to \mathrm{B})^\alpha$ & $\gamma = \alpha\cdot\beta$. If $\delta \in 2^{\mathbb{N}}$ & A^δ then $\mathrm{B}^{\alpha\cdot\delta}$ by definition of $(\mathrm{A} \to \mathrm{B})^\alpha$, and $\delta\cdot\gamma = (\alpha\cdot\delta)\cdot\beta$ so $\mathrm{B}^{\delta\cdot\gamma}$ by the induction hypothesis on B. So $(\mathrm{A} \to \mathrm{B})^\gamma$.

If E is $\neg\mathrm{A}$ where A satisfies the lemma, assume $\alpha, \beta, \gamma \in 2^{\mathbb{N}}$ & $(\neg\mathrm{A})^\alpha$ & $\gamma = \alpha\cdot\beta$. If $\delta \in 2^{\mathbb{N}}$ & A^δ, then $\mathrm{Z}(\alpha\cdot\delta)$ by definition of $(\neg\mathrm{A})^\alpha$, so $\mathrm{Z}(\gamma\cdot\delta)$ by Proposition 2.2. So $(\neg\mathrm{A})^\gamma$.

2.4. **Lemma.** $\mathbf{IA}_1 \vdash \forall \beta \in 2^{\mathbb{N}}(\mathrm{Z}(\beta) \to \mathrm{E}^\beta)$ for all formulas E.

2.5. **Lemma.**

(a) $\mathbf{IA}_1 \vdash \forall \alpha \in 2^{\mathbb{N}}((\mathrm{A} \to \mathrm{B})^\alpha \to (\mathrm{A}^\alpha \to \mathrm{B}^\alpha))$.
(b) $\mathbf{IA}_1 \vdash \forall \alpha \in 2^{\mathbb{N}}(\mathrm{A} \to \mathrm{B})^\alpha \leftrightarrow \forall \alpha \in 2^{\mathbb{N}}(\mathrm{A}^\alpha \to \mathrm{B}^\alpha)$.

Proofs. (a) follows immediately from the definition with the fact that $\alpha\cdot\alpha = \alpha$ for all $\alpha \in 2^{\mathbb{N}}$.

For (b), the implication from left to right follows from (a) by logic. For the implication from right to left assume $\forall \alpha \in 2^{\mathbb{N}}(\mathrm{A}^\alpha \to \mathrm{B}^\alpha)$ and $\alpha, \beta \in 2^{\mathbb{N}}$ and A^β; then $\mathrm{A}^{\alpha\cdot\beta}$ by Lemma 2.3, so $\mathrm{B}^{\alpha\cdot\beta}$ by the assumption. So $(\mathrm{A} \to \mathrm{B})^\alpha$.

2.6. **Lemma.** If E is $\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$ (i.e., $\mathrm{Z}(\alpha)$) then $\mathbf{IA}_1$ proves:

(a) $\forall \beta \in 2^{\mathbb{N}}(\mathrm{E}^\beta \leftrightarrow \mathrm{E} \vee \mathrm{Z}(\beta))$.
(b) $\forall \beta \in 2^{\mathbb{N}}((\neg\mathrm{E})^\beta \leftrightarrow (\mathrm{E} \to \mathrm{Z}(\beta)))$.
(c) $\forall \beta \in 2^{\mathbb{N}}((\neg\neg\mathrm{E})^\beta \leftrightarrow \mathrm{E} \vee \mathrm{Z}(\beta))$.
(d) $\forall \beta \in 2^{\mathbb{N}}(\neg\neg\mathrm{E} \leftrightarrow \mathrm{E})^\beta$.

[2]This simplifying convention, which will be used extensively in what follows, could be avoided by replacing $\mathrm{B}^{\alpha\cdot\beta}$ by $\forall \gamma \in 2^{\mathbb{N}}(\gamma = \alpha\cdot\beta \to \mathrm{B}^\gamma)$, as in the statement of Lemma 2.3.

Proofs. (a) is immediate by Definition 2.1 with intuitionistic logic. For (b), under the assumption $\beta \in 2^{\mathbb{N}}$ and using (a), Proposition 2.2, intuitionistic logic and the fact that $\beta \cdot \beta = \beta$, we have the following chain of equivalences:

$$(\neg \mathrm{E})^\beta \leftrightarrow \forall \gamma \in 2^{\mathbb{N}}(\mathrm{E}^\gamma \to \mathrm{Z}(\beta \cdot \gamma))$$
$$\leftrightarrow \forall \gamma \in 2^{\mathbb{N}}(\mathrm{E} \vee \mathrm{Z}(\gamma) \to \mathrm{Z}(\beta \cdot \gamma))$$
$$\leftrightarrow \forall \gamma \in 2^{\mathbb{N}}(\mathrm{E} \to \mathrm{Z}(\beta \cdot \gamma)) \leftrightarrow (\mathrm{E} \to \mathrm{Z}(\beta)).$$

For (c), under the assumption $\beta \in 2^{\mathbb{N}}$, by (b) we have

$$(\neg\neg \mathrm{E})^\beta \leftrightarrow \forall \gamma \in 2^{\mathbb{N}}((\neg \mathrm{E})^\gamma \to \mathrm{Z}(\beta \cdot \gamma)) \leftrightarrow \forall \gamma \in 2^{\mathbb{N}}((\mathrm{E} \to \mathrm{Z}(\gamma)) \to \mathrm{Z}(\beta \cdot \gamma)).$$

If $\forall \gamma \in 2^{\mathbb{N}}((\mathrm{E} \to \mathrm{Z}(\gamma)) \to \mathrm{Z}(\beta \cdot \gamma))$, let $\gamma = \lambda \mathrm{x}.\mathrm{sg}(\alpha(\mathrm{x}))$; then $\mathrm{Z}(\gamma) \leftrightarrow \mathrm{Z}(\alpha)$ and $\gamma \in 2^{\mathbb{N}}$. Then $(\mathrm{Z}(\gamma) \to \mathrm{Z}(\gamma)) \to \mathrm{Z}(\beta \cdot \gamma)$ since E is $\mathrm{Z}(\alpha)$, so $\mathrm{Z}(\beta \cdot \gamma)$, so $\mathrm{Z}(\beta) \vee \mathrm{Z}(\gamma)$ by Proposition 2.2, so $\mathrm{Z}(\beta) \vee \mathrm{E}$, so $\mathrm{E} \vee \mathrm{Z}(\beta)$. For the converse use Proposition 2.2. Then (d) follows from (a) and (c) with Lemma 2.5(b).

2.7. **Lemma.**

(a) If E has no $\to$ or $\neg$ then $\mathbf{IA}_1 \vdash (\mathrm{E}^{\lambda \mathrm{z}.1} \leftrightarrow \mathrm{E})$.
(b) $\mathbf{IA}_1 \vdash (\neg \mathrm{A})^{\lambda \mathrm{z}.1} \to \neg(\mathrm{A}^{\lambda \mathrm{z}.1})$ for all formulas A.
(c) If E is constructed from prime formulas and their negations using only & and $\vee$, then $\mathbf{IA}_1 \vdash (\mathrm{E}^{\lambda \mathrm{z}.1} \to \mathrm{E})$.

3. Applications to Subsystems of Kleene's Formal System I for Intuitionistic Analysis

3.1. **Theorem.** If E is derivable in $\mathbf{IA}_1$ from assumptions $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{k}, \mathrm{F}_{\mathrm{k}+1}, \ldots, \mathrm{F}_\mathrm{m}$ such that $\forall \beta \in 2^{\mathbb{N}}\, (\mathrm{F}_\mathrm{i})^\beta$ is derivable in $\mathbf{IA}_1$ from $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{k}$ for each $1 \leq i \leq m$, with all free variables held constant in the deductions, then $\forall \beta \in 2^{\mathbb{N}}\, \mathrm{E}^\beta$ is also derivable in $\mathbf{IA}_1$ from $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{k}$ with all free variables held constant.

Proof. $\mathbf{IA}_1 \vdash \forall \alpha \in 2^{\mathbb{N}}\, \mathrm{E}^\alpha$ when E is any axiom of $\mathbf{IA}_1$, using the lemmas in the previous section with $\forall \alpha \in 2^{\mathbb{N}}(\alpha \cdot \alpha = \alpha)$ as appropriate (e.g. for the mathematical induction schema). If $\forall \beta \in 2^{\mathbb{N}}\, \mathrm{A}^\beta$ and $\forall \beta \in 2^{\mathbb{N}}\, (\mathrm{A} \to \mathrm{B})^\beta$ are derivable in $\mathbf{IA}_1$ from $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{k}$ with the free variables held constant, then by Lemma 2.5(b) so is $\forall \beta \in 2^{\mathbb{N}}\, \mathrm{B}^\beta$; and similarly for the other rules of inference.

3.2. **Lemma.** $\mathbf{IA}_1 + \mathrm{AC}_{00} \vdash \forall \beta \in 2^{\mathbb{N}}(\mathrm{AC}_{00})^\beta$, and similarly for qf-$\mathrm{AC}_{00}$, AC_{01}.

Proofs. By the definition with Lemma 2.5(b).

3.3. **Lemma.** $\mathbf{IA}_1 + \mathrm{BI}_1 \vdash \forall \beta \in 2^{\mathbb{N}}(\mathrm{BI}_1)^\beta$ where BI_1 is the bar induction schema

$$\forall \alpha \exists \mathrm{x} \rho(\overline{\alpha}(\mathrm{x})) = 0 \ \& \ \forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \ \& \ \rho(\mathrm{w}) = 0 \to \mathrm{A}(\mathrm{w}))$$
$$\& \ \forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \ \& \ \forall \mathrm{s} \mathrm{A}(\mathrm{w} * \langle \mathrm{s}+1 \rangle) \to \mathrm{A}(\mathrm{w})) \to \mathrm{A}(\langle\ \rangle).$$

Proof. Assume $\beta \in 2^{\mathbb{N}}$ and

(i) $(\forall \alpha \exists \mathrm{x} \rho(\overline{\alpha}(\mathrm{x})) = 0)^\beta$,
(ii) $(\forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \ \& \ \rho(\mathrm{w}) = 0 \to \mathrm{A}(\mathrm{w})))^\beta$,
(iii) $(\forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \ \& \ \forall \mathrm{s} \mathrm{A}(\mathrm{w} * \langle \mathrm{s}+1 \rangle) \to \mathrm{A}(\mathrm{w})))^\beta$.

By Lemma 2.5 it will be enough to prove $\mathrm{A}^\beta(\langle\ \rangle)$. By the definition and the lemmas in the previous section, over $\mathbf{IA}_1$ the numbered assumptions are equivalent respectively to

(i') $\forall \alpha \exists \mathrm{x}(\rho(\overline{\alpha}(\mathrm{x})) = 0 \vee \mathrm{Z}(\beta))$,

(ii') $\forall w \forall \gamma \in 2^{\mathbb{N}}((\mathrm{Seq}(w) \;\&\; \rho(w) = 0) \vee Z(\gamma) \to A^{\beta\cdot\gamma}(w)))$,
(iii') $\forall w \forall \gamma \in 2^{\mathbb{N}}(\mathrm{Seq}(w) \;\&\; \forall s A^{\gamma}(w * \langle s+1 \rangle) \to A^{\beta\cdot\gamma}(w))$.

In $\mathbf{IA}_1$ we may define $\sigma \in 2^{\mathbb{N}}$ so that

$$\sigma(w) = 0 \leftrightarrow \rho(w) = 0 \vee \exists x \leq w \beta(x) = 0.$$

From (i') it follows immediately that $\forall \alpha \exists x \sigma(\overline{\alpha}(x) = 0)$. From (ii') with $\gamma = \beta$ and the fact that $\beta = \beta \cdot \beta$ we have $\forall w(\mathrm{Seq}(w) \;\&\; \sigma(w) = 0 \to A^{\beta}(w))$. From (iii') similarly, $\forall w(\mathrm{Seq}(w) \;\&\; \forall s A^{\beta}(w * \langle s+1 \rangle) \to A^{\beta}(w))$, so $A^{\beta}(\langle\,\rangle)$ follows by BI_1.

3.4. **Lemma.** $\mathbf{IA}_1 + \mathrm{CC}_{10} \vdash \forall \beta \in 2^{\mathbb{N}}\, (\mathrm{CC}_{10})^{\beta}$ where CC_{10} is

$$\forall \alpha \exists x A(\alpha, x) \to \exists \sigma \forall \alpha (\exists y \sigma(\overline{\alpha}(y)) > 0 \;\&\; \forall y(\sigma(\overline{\alpha}(y)) > 0 \to A(\alpha, \sigma(\overline{\alpha}(y) \dot{-} 1)))),$$

which is equivalent over $\mathbf{IA}_1$ + qf-AC_{00} to Kleene and Vesley's continuous choice schema *27.2 ("Brouwer's Principle for numbers").

Proof. Assume $\beta \in 2^{\mathbb{N}}$ and $\forall \alpha \exists x A^{\beta}(\alpha, x)$. By Lemma 2.5(b) it will be enough to find a σ such that for all α:

(i) $\exists y(\sigma(\overline{\alpha}(y)) > 0 \vee Z(\beta))$ and
(ii) $\forall y \forall \gamma \in 2^{\mathbb{N}}(\sigma(\overline{\alpha}(y)) > 0 \vee Z(\gamma) \to A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y) \dot{-} 1)))$.

CC_{10} provides a σ such that for all α:

(i') $\exists y\ \sigma(\overline{\alpha}(y)) > 0$ and
(ii') $\forall y(\sigma(\overline{\alpha}(y)) > 0 \to A^{\beta}(\alpha, \sigma(\overline{\alpha}(y) \dot{-} 1)))$.

Obviously (i') entails (i). To prove (ii), let $y \in \mathbb{N}$ and $\gamma \in 2^{\mathbb{N}}$. If $\sigma(\overline{\alpha}(y)) > 0$ then $A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y) \dot{-} 1))$ by (ii') with Lemma 2.3, and if $Z(\gamma)$ then $A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y) \dot{-} 1))$ by Lemma 2.4, so $\sigma(\overline{\alpha}(y)) > 0 \vee Z(\gamma) \to A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y) \dot{-} 1))$.

3.5. **Lemma.** $\mathbf{IA}_1 + \mathrm{CC}_{11} \vdash \forall \beta \in 2^{\mathbb{N}}\, (\mathrm{CC}_{11})^{\beta}$ where CC_{11} is the corresponding equivalent of "Brouwer's Principle for functions" (axiom schema ${}^{x}27.1$ of [5]):

$$\forall \alpha \exists \beta A(\alpha, \beta) \to \exists \sigma \forall \alpha [\forall x \exists y \sigma(\langle x+1 \rangle * \overline{\alpha}(y)) > 0 \;\&\; \forall \beta(\forall x \exists y \sigma(\langle x+1 \rangle * \overline{\alpha}(y)) = \beta(x) + 1 \to A(\alpha, \beta))].$$

The proof is similar to that for CC_{10} and will be omitted.

3.6. **Corollary.** If $\mathbf{T}$ is $\mathbf{IA}_1$, Kleene's neutral theory $\mathbf{B} = \mathbf{IA}_1 + \mathrm{AC}_{01} + \mathrm{BI}_1$, $\mathbf{I} = \mathbf{B} + \mathrm{CC}_{11}$ or any subsystem of $\mathbf{I}$ obtained by adding to $\mathbf{IA}_1$ any of the schemas qf-AC_{00}, AC_{00}, AC_{01}, BI_1 and/or CC_{10}, then $\mathbf{T} + \mathrm{MP}_1$ and $\mathbf{T}$ prove the same Π^0_2 statements.

Proof. By Lemma 2.6(d), $\mathbf{T} \vdash \forall \beta \in 2^{\mathbb{N}}(\mathrm{MP}_1)^{\beta}$. Hence by Theorem 3.1 with Lemmas 3.2 - 3.5, if $\mathbf{T} + \mathrm{MP}_1 \vdash E$ then $\mathbf{T} \vdash \forall \beta \in 2^{\mathbb{N}}\, E^{\beta}$.

If E is $\forall x \exists y A(x, y)$ where $A(x, y)$ has only bounded numerical quantifiers, then $A(x, y)$ is equivalent over $\mathbf{IA}_1$ to a formula of the type described in Lemma 2.7(c), so by Theorem 3.1 and Lemma 2.5(b): if $\mathbf{T} \vdash E^{\lambda z.1}$ then $\mathbf{T} \vdash E$.

3.7. **Remarks.** Lemma 2.7(c) holds also for formulas E constructed from prime formulas and their negations using only $\&, \vee, \forall$ and $\exists$, in particular for all prenex formulas. It follows, for each subsystem $\mathbf{T}$ of Kleene's $\mathbf{I}$ described in the statement of Corollary 3.6, that any prenex formula provable in $\mathbf{T} + \mathrm{MP}_1$ is provable in $\mathbf{T}$.

The question whether or not the "minimal" system $\mathbf{M} = \mathbf{IA}_1 + \mathrm{AC}_{00}!$ (which Kleene used in [4] to formalize the theory of recursive funcionals) proves the same Π^0_2 formulas as $\mathbf{M} + \mathrm{MP}_1$ is still open, as far as we know. Because the translation

$\mathrm{E} \mapsto \mathrm{E}^{\beta}$ essentially involves (binary) sequence quantifiers, the corresponding question is still open for $\mathbf{IA}_1 + \mathrm{AC}_{00}^{Ar}$ and for Solovay's system $\mathbf{S} = \mathbf{IA}_1 + \mathrm{AC}_{00}^{Ar} + \mathrm{BI}_1$, where AC_{00}^{Ar} is the restriction of AC_{00} to arithmetical formulas $\mathrm{A}(\mathrm{x}, \mathrm{y})$ (with sequence parameters allowed). In the presence of bar induction, weak choice axioms interact with MP_1 in sometimes surprising ways; for example, Solovay proved that $\mathbf{S}$ can be interpreted negatively in $\mathbf{IA}_1 + \mathrm{BI}_1 + \mathrm{MP}_1$.[3]

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[3]In fact, his proof justifies a stronger result: $\mathbf{S}$ can be interpreted negatively in $\mathbf{IA}_1 + \mathrm{BI}_1 + \mathrm{DNS}_1$, where DNS_1 is the double negation shift principle $\forall\alpha\neg\neg\exists \mathrm{x} \mathrm{A}(\overline{\alpha}(\mathrm{x})) \rightarrow \neg\neg\forall\alpha\exists \mathrm{x} \mathrm{A}(\overline{\alpha}(\mathrm{x}))$ for quantifier-free formulas $\mathrm{A}(\mathrm{w})$.