MARKOV’S PRINCIPLE AND SUBSYSTEMS OF
INTUITIONISTIC ANALYSIS

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ABSTRACT. Using a technique developed by Coquand and Hofmann [3] we verify that adding the analytical form $\text{MP}_1$: $\forall \alpha (\neg \neg \exists x \alpha(x) = 0 \rightarrow \exists x \alpha(x) = 0)$ of Markov’s Principle does not increase the class of $\Pi_0^2$ formulas provable in Kleene and Vesley’s formal system for intuitionistic analysis, or in subsystems obtained by omitting or restricting various axiom schemas in specified ways.

INTRODUCTION

In [6] Kleene proved that Markov’s Principle $\text{MP}_1$ is neither provable nor refutable in his formal system $I$ for intuitionistic analysis. By the Friedman-Dragalin translation, Markov’s Rule is admissible for $I$ and many subsystems.

We show that adding $\text{MP}_1$ as an axiom to $I$ does not increase consistency strength, in the sense that no additional $\Pi_0^2$ formulas become provable. The method, adapted from Coquand and Hofmann’s dynamic modification [3] of the Friedman-Dragalin translation, works also for subsystems of $I$ with a few interesting exceptions.

1. LANGUAGE, LOGIC, AND BASIC MATHEMATICAL AXIOMS

1.1. The two-sorted formal language and intuitionistic predicate logic. Kleene and Vesley’s language $\mathcal{L}_1$ for two-sorted intuitionistic number theory or “intuitionistic analysis” has variables $a, b, c, \ldots, x, y, z, \ldots$, intended to range over natural numbers; variables $\alpha, \beta, \gamma, \ldots$, intended to range over one-place number-theoretic functions (choice sequences); finitely many constants $0', +, f_4, \ldots, f_p$, each representing a primitive recursive function or functional, where $f_i$ has $k_i$ places for number arguments and $l_i$ places for type-1 function arguments; parentheses indicating function application; and Church’s $\lambda$.

The terms (of type 0) and functors (of type 1) are defined inductively as follows. The number variables and 0 are terms. The function variables and each $f_i$ with $k_i = 1, l_i = 0$ are functors. If $t_1, \ldots, t_{k_i}$ are terms and $u_1, \ldots, u_{l_i}$ are functors, then $f_i(t_1, \ldots, t_{k_i}, u_1, \ldots, u_{l_i})$ is a term. If $x$ is a number variable and $t$ is a term, then $\lambda x.t$ is a functor. And if $u$ is a functor and $t$ is a term, then $(u)(t)$ is a term.

There is one relation symbol $=$ for equality between terms; equality between functors $u, v$ is defined extensionally by $u = v \equiv \forall x(u(x) = v(x))$ (where $x$ is not free in $u$ or $v$). The atomic formulas of $\mathcal{L}_1$ are the expressions $s = t$ where $s, t$ are terms. Composite formulas are defined inductively, using the connectives $\&$, $\lor$, $\rightarrow$, $\neg$, quantifiers $\forall$, $\exists$ of both sorts, and parentheses (often omitted under the usual conventions on scope). $A \leftrightarrow B$ is defined by $(A \rightarrow B) \& (B \rightarrow A)$.

The logical axioms and rules are those of two-sorted intuitionistic predicate logic, as presented in [6] (building on [4]). If the intuitionistic axiom schema
\[ \neg A \rightarrow (A \rightarrow B) \] were replaced by \( \neg \neg A \rightarrow A \) (of which Markov’s Principle \( \text{MP}_1 \) is a special case), two-sorted classical predicate logic would result.

1.2. Two-sorted intuitionistic arithmetic \( \text{IA}_1 \). This is a conservative extension, in the language \( \mathcal{L}_1 \), of the first-order intuitionistic arithmetic \( \text{IA}_0 \) in [4] based on 0′, +, \( \cdot \). The mathematical axioms of \( \text{IA}_1 \) are:

(a) The axiom-schema of mathematical induction (for all formulas of \( \mathcal{L}_1 \)):
\[ A(0) \land \forall x (A(x) \rightarrow A(x')) \rightarrow A(x). \]

(b) The axioms of \( \text{IA}_0 \) for 0′, +, \( \cdot \) (axioms 14-21 on page 82 of [4]) and the axioms expressing the primitive recursive definitions of the additional function constants \( f_4, \ldots, f_{20} \) given in [6] and [5].

(c) The open equality axiom: \( x = y \rightarrow \alpha(x) = \alpha(y) \).

(d) The axiom-schema of \( \lambda \)-conversion: \( (\lambda x. t(x))(s) = t(s) \), where \( t(x) \) is a term and \( s \) is free for \( x \) in \( t(x) \).

For readers familiar with [6], \( \text{IA}_1 \) is the subsystem of the “basic system” \( \mathcal{B} \) obtained by omitting the axiom schemas of countable choice and bar induction (\( \exists 2.1 \) and \( \exists 26.3 \), respectively).

In addition to the open equality axiom (c), the equality axioms
\[ \alpha_1 = \beta_1 \land \ldots \land \alpha_i = \beta_i \rightarrow f_i(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_i) = f_i(x_1, \ldots, x_k, \beta_1, \ldots, \beta_i), \]
are provable for all function constants \( f_i \). Thus \( \text{IA}_1 \) satisfies the replacement property of equality for functors as well as for terms.

\( \text{IA}_1 \) can only prove the existence of primitive recursive sequences, in the sense that each closed theorem of the form \( \exists \alpha A(\alpha) \) has a primitive recursive witness. The finite list of primitive recursive function constants, with their corresponding axioms, is intended to be expanded as needed. Here we use the \( \lambda \) notation to explicitly define termwise multiplication of sequences: \( (\alpha \cdot \beta) \) will abbreviate \( \lambda x. (\alpha(x) \cdot \beta(x)) \). We also define \( \text{sg}(\alpha) = \lambda x. \text{sg}(\alpha(x)) \), in effect adding binary sequence variables to \( \mathcal{L}_1 \).

1.3. Intuitionistic recursive analysis \( \text{IRA} \). The principle of countable choice for numbers is expressed in \( \mathcal{L}_1 \) by the schema (\( \exists 2.2 \) in [6]):
\[ \text{AC}_{00} : \forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)), \]

\( \text{AC}_{00} \)

1\footnote{\( f_0 - f_3 \) are 0′, +, \( \cdot \), respectively. \( f_4 - f_{20} \) represent the primitive recursive function \( x \Rightarrow y \) (exponentiation), \( f_{21} \) represents the prime factorization of \( x \).}

\( f_{21}(a) \) represents the number of positive exponents in the prime factorization of \( a \). Bounded quantifiers are defined with the help of bounded sum and product. \( \text{Seq}(\alpha) \) is a prime formula equivalent to \( a > 0 \land \forall i < \text{lh}(a) (\alpha_i) \), \( a > 0 \), expressing “\( a \) codes the finite sequence \( \langle (a_0 - 1, \ldots, a_{\text{lh}(a) - 1} \rangle \).”

\( f_{22}(a, b) = a \ast b \) produces a code for the concatenation of two finite sequences from their codes.

\( \langle \rangle \) is the empty sequence, and \( f_{23}(x, \alpha) = \Sigma(x) = \Pi_{i < \text{lh}(\alpha)} \) represents the standard code \( (\alpha(0) + 1, \ldots, \alpha(x - 1) + 1) \) for the \( x^{\text{th}} \) initial segment of \( \alpha \). This coding is not onto \( \mathbb{N} \), but it satisfies \( \langle a_0 + 1, \ldots, a_k + 1 \rangle \ast \langle a_{k+1} + 1, \ldots, a_{m+1} \rangle = \langle a_0 + 1, \ldots, a_m + 1 \rangle \). In contrast, \( f_{24}(x, \alpha) = \bar{\alpha}(x) = \Pi_{i < \text{lh}(\alpha)} \) cannot code finite sequences directly as \( \langle a_0, \ldots, a_k \rangle = \langle a_0, \ldots, a_k \rangle \).

\( f_{25}(a, b) = a \ast b = \Pi_{i < \text{max}(a, b)} \) represents the course-of-values function for the characteristic function of the predicate “\( y \) is a computation tree number.” These suffice for Kleene’s formal treatment (\[5\] Part I) of recursive partial functionals, including the recursion theorem and a normal form theorem.
where $\alpha, x$ must be free for $y$ in $A(x, y)$. Intuitionistic recursive analysis IRA can be axiomatized, as a subsystem of Kleene and Vesley’s $B$, by $IA_1 + qf-AC_{00}$, where $qf-AC_{00}$ is the restriction of $AC_{00}$ to formulas $A(x, y)$ without sequence quantifiers and with only bounded number quantifiers. IRA ensures that the range of the type-1 variables contains all general recursive sequences and is closed under general recursive processes. Troelstra’s EL and Veldman’s BIM are alternative axiomatizations of IRA, cf. [8], [7].

In the two-sorted language, IRA + MP$_1$ + CT$_1$ formalizes Russian recursive analysis (RUSS in [2]), where MP$_1$ is the functional form of Markov’s Principle

$$MP_1 : \forall \alpha[\neg \exists x \alpha(x) = 0 \rightarrow \exists x \alpha(x) = 0]$$

and CT$_1$ expresses Church’s Thesis:

$$CT_1 : \forall \alpha \exists e \forall x \exists y [T_0(e, x, y) \land U(y) = \alpha(x)].$$

The general recursive functions form a classical ω-model of RUSS and hence of IRA, but RUSS + AC$_{00}$ (unlike IRA + AC$_{00}$) is inconsistent with classical logic.

2. Definition of the Translation, and Properties Proved in $IA_1$

2.1. Definition. Let $Z(\alpha)$ abbreviate $\exists x \alpha(x) = 0$. To each formula $E$ of $L_1$ and each sequence variable $\alpha$ not occurring in $E$, we associate another formula $E^\alpha$ with the same free variables plus $\alpha$, by induction on the logical form of $E$ as follows. For cases 4 and 5, $\beta$ should be distinct from $\alpha$, and $A^{(\beta)}$ is the result of substituting $sg(\beta)$ for $\gamma$ in the definition of $A^\gamma$. Similarly for $B^{\alpha, \beta}$ in Case 4.

1. $P^\alpha$ is $P \lor Z(\alpha)$ if $P$ is prime.
2. $(A \land B)^\alpha$ is $A^\alpha \land B^\alpha$.
3. $(A \lor B)^\alpha$ is $A^\alpha \lor B^\alpha$.
4. $(A \rightarrow B)^\alpha$ is $\forall \beta(A^{(\beta)} \rightarrow B^{\alpha, \beta})$.
5. $(\neg A)^\alpha$ is $\forall \beta(A^{(\beta)} \rightarrow Z(\alpha \cdot \beta))$.
6. $(\forall x A(x))^\alpha$ is $\forall x A^\alpha(x)$.
7. $(\exists x A(x))^\alpha$ is $\exists x A^\alpha(x)$.
8. $(\forall \gamma A(\gamma))^\alpha$ is $\forall \gamma A^\alpha(\gamma)$.
9. $(\exists \gamma A(\gamma))^\alpha$ is $\exists \gamma A^\alpha(\gamma)$.

From now on, let $\alpha \in 2^0$ abbreviate $\alpha = sg(\alpha)$.

2.2. Proposition.

(a) $IA_1 \vdash \forall \alpha [\forall \beta (Z(\alpha \cdot \beta) \leftrightarrow Z(\alpha) \lor Z(\beta))].$

(b) $IA_1 \vdash \forall \alpha (E^\alpha \leftrightarrow E^{(sg(\alpha))})$ for all formulas $E$.

(c) $IA_1 \vdash \forall \alpha \in 2^0 (E(\alpha) \leftrightarrow E^{(sg(\alpha)))}.$

Proofs. (a) holds by intuitionistic logic, (b) is proved by formula induction, and the replacement property of equality for functors guarantees (c).

2.3. Lemma. $IA_1 \vdash \forall \alpha \forall \beta \forall \gamma (E^\alpha \land \gamma = \alpha \cdot \beta \rightarrow E^\gamma)$.

Proof. Only Cases 4 and 5 require attention. If $E$ is $A \rightarrow B$ where $A, B$ both satisfy the lemma, assume $(A \rightarrow B)^\alpha \land \gamma = \alpha \cdot \beta$. If $A^{(\beta)}$ then $B^{\alpha, \beta}$ by definition of $(A \rightarrow B)^\alpha$, and $\delta \cdot \gamma = (\alpha \cdot \delta) \cdot \beta$ so $B^{\alpha, \gamma}$ by the induction hypothesis on $B$. So $(A \rightarrow B)^\gamma$.

If $E$ is $\neg A$ where $A$ satisfies the lemma, assume $(\neg A)^\alpha \land \gamma = \alpha \cdot \beta$. If $A^{(\beta)}$, then $Z(\alpha \cdot \delta)$ by definition of $(\neg A)^\alpha$, so $Z(\gamma \cdot \delta)$ by Proposition 2.2(a). So $(\neg A)^\gamma$.

2.4. Lemma. $IA_1 \vdash \forall \alpha (Z(\alpha) \rightarrow E^\alpha)$ for all formulas $E$. 

MARKOV’S PRINCIPLE AND SUBSYSTEMS OF INTUITIONISTIC ANALYSIS 3
2.5. Lemma.
(a) $\text{IA}_1 \vdash \forall \alpha \in 2^N((A \rightarrow B)^\alpha \rightarrow (A^\alpha \rightarrow B^\alpha))$.
(b) $\text{IA}_1 \vdash \forall \alpha \in 2^N(A \rightarrow B)^\alpha \iff \forall \alpha \in 2^N(A^\alpha \rightarrow B^\alpha)$.

Proofs. (a) follows immediately from the definition and Proposition 2.2(b) with the fact that $\alpha \cdot \alpha = \alpha$ for all $\alpha \in 2^N$.

For (b), the implication from left to right follows from (a) by logic. For the converse assume $\forall \alpha \in 2^N(A^\alpha \rightarrow B^\alpha)$ and $\alpha \in 2^N$ and $A^\alpha \beta$; then $B^\beta$ by the assumption, so $B^\beta$ by Proposition 2.2(b), so $B^\alpha \cdot \beta$ by Lemma 2.3. So $(A \rightarrow B)^\alpha$.

2.6. Lemma. If $E$ is $\exists \alpha(x) = 0$ (i.e., $Z(\alpha)$) then $\text{IA}_1$ proves:
(a) $\forall \beta \in 2^N(E^\beta \leftrightarrow E \vee Z(\beta))$.
(b) $\forall \beta \in 2^N((-E)^\beta \leftrightarrow (E \rightarrow Z(\beta)))$.
(c) $\forall \beta \in 2^N((-E)^\beta \leftrightarrow (E \vee Z(\beta)))$.
(d) $\forall \beta \in 2^N(-E \leftrightarrow E)^\beta$.

Proofs. (a) is immediate by Definition 2.1 with intuitionistic logic. For (b), under the assumption $\beta \in 2^N$ and using (a), Proposition 2.2, intuitionistic logic and the fact that $\beta \cdot \beta = \beta$, we have the following chain of equivalences:

$(-E)^\beta \leftrightarrow \forall \gamma \in 2^N(E^\gamma \rightarrow Z(\beta \cdot \gamma))$
$
\leftrightarrow \forall \gamma \in 2^N(E \vee Z(\gamma) \rightarrow Z(\beta \cdot \gamma))$
$
\leftrightarrow \forall \gamma \in 2^N(E \rightarrow Z(\beta \cdot \gamma)) \leftrightarrow (E \rightarrow Z(\beta))$.

For (c), under the assumption $\beta \in 2^N$, by (b) we have

$(-E)^\beta \leftrightarrow \forall \gamma \in 2^N((-E)^\gamma \rightarrow Z(\beta \cdot \gamma)) \leftrightarrow \forall \gamma \in 2^N((E \rightarrow Z(\gamma)) \rightarrow Z(\beta \cdot \gamma))$.

If $\forall \gamma \in 2^N((E \rightarrow Z(\gamma)) \rightarrow Z(\beta \cdot \gamma))$, let $\gamma = \text{sg}(\alpha)$; then $Z(\gamma) \leftrightarrow Z(\alpha)$ and $\gamma \in 2^N$. Then $(Z(\gamma) \rightarrow Z(\gamma)) \rightarrow Z(\beta \cdot \gamma)$ since $E$ is $Z(\alpha)$, so $Z(\beta \cdot \gamma)$, so $Z(\beta) \vee Z(\gamma)$ by Proposition 2.2(a), so $Z(\beta) \vee E$, so $E \vee Z(\beta)$. For the converse use Proposition 2.2(a). Then (d) follows from (a) and (c) with Lemma 2.5(b).

2.7. Lemma.
(a) If $E$ has no $\rightarrow$ or $\neg$ then $\text{IA}_1 \vdash (E^{\lambda x.1} \leftrightarrow E)$.
(b) $\text{IA}_1 \vdash (\neg A)^{\lambda x.1} \rightarrow (\neg (A^{\lambda x.1}))$ for all formulas $A$.
(c) If $E$ is constructed from prime formulas and their negations using only $\&$ and $\vee$, then $\text{IA}_1 \vdash (E^{\lambda x.1} \rightarrow E)$.

3. Applications to Subsystems of Kleene’s Formal System I for Intuitionistic Analysis

3.1. Theorem. If $T$ is a theory extending $\text{IA}_1$ by axioms and axiom schemas $F_1, \ldots, F_n$ such that $T \vdash \forall \alpha \in 2^N(F_i)^\alpha$ for $i = 1, \ldots, n$, and if $E$ is derivable in $T$ from assumptions $A_1, \ldots, A_m$ with all free variables held constant in the deduction, then $E^\beta$ is derivable in $T$ from the assumptions $\beta \in 2^N$, $(A_1)^{\beta}, \ldots, (A_m)^{\beta}$ with all free variables held constant.

Proof. $\text{IA}_1 \vdash \forall \alpha \in 2^N E^\alpha$ when $E$ is any axiom of $\text{IA}_1$, using the lemmas in the previous section with $\forall \alpha \in 2^N(\alpha \cdot \alpha = \alpha)$ as appropriate (e.g. for the mathematical induction schema). If $B^\beta$ and $(B \rightarrow C)^\beta$ are derivable in $\text{IA}_1$ from $\beta = \text{sg}(\beta)$, $(A_1)^{\beta}, \ldots, (A_m)^{\beta}$ with the free variables held constant, then by Lemma 2.5(a) so is $B^\beta \rightarrow C^\beta$, and therefore also $C^\beta$. Similarly for the other rules of inference.
3.2. Lemma. $\textbf{IA}_1 + AC_{00} \vdash \forall \beta \in 2^N(AC_{00})^\beta$, and similarly for $qf-AC_{00}$, $AC_{01}$.

Proofs. By the definition with Lemma 2.5(b).

3.3. Lemma. $\textbf{IA}_1 + BI_1 \vdash \forall \beta \in 2^N(BI_1)^\beta$ where $BI_1$ is the bar induction schema

\[
\forall \alpha \exists x \rho(\overline{\alpha}(x)) = 0 \& \forall w(\text{Seq}(w) \& \rho(w) = 0 \rightarrow A(w)) \& \forall w(\text{Seq}(w) \& \forall sA(w * (s + 1)) \rightarrow A(w)) \rightarrow A(\langle \rangle).
\]

Proof. Assume $\beta \in 2^N$ and

(i) $\langle \forall \alpha \exists x \rho(\overline{\alpha}(x)) = 0 \rangle^\beta$,

(ii) $\forall w(\text{Seq}(w) \& \rho(w) = 0 \rightarrow A(w))^\beta$,

(iii) $\forall w(\text{Seq}(w) \& \forall sA(w * (s + 1)) \rightarrow A(w))^\beta$.

By Lemma 2.5 it will be enough to prove $A^\beta(\langle \rangle)$. By the definition and the lemmas in the previous section, over $\textbf{IA}_1$ the numbered assumptions are equivalent respectively to

(i’) $\forall \alpha \exists x(\rho(\overline{\alpha}(x)) = 0 \lor Z(\beta))$,

(ii’) $\forall w(\text{Seq}(w) \& \rho(w) = 0 \lor Z(\gamma) \rightarrow A^\beta(\overline{\gamma}(w)))$,

(iii’) $\forall w(\text{Seq}(w) \& \forall sA^\gamma(w * (s + 1)) \rightarrow A^\beta(\overline{\gamma}(w))$.

In $\textbf{IA}_1$ we may define $\sigma \in 2^N$ so that $\sigma(w) = 0 \leftrightarrow \rho(w) = 0 \lor \exists x \leq w \beta(x) = 0$.

From (i’) it follows immediately that $\forall \alpha \exists x \sigma(\overline{\alpha}(x)) = 0$. From (ii’) with $\gamma = \beta$ and the fact that $\beta = \beta \cdot \beta$ we have $\forall w(\text{Seq}(w) \& \sigma(w) = 0 \rightarrow A^\beta(w))$. From (iii’) similarly, $\forall w(\text{Seq}(w) \& \forall sA^\gamma(w * (s + 1)) \rightarrow A^\beta(\overline{\gamma}(w))$, so $A^\beta(\langle \rangle)$ follows by $BI_1$.

3.4. Lemma. $\textbf{IA}_1 + CC_{10} \vdash \forall \beta \in 2^N(CC_{10})^\beta$ where $CC_{10}$ is

\[
\forall \alpha \exists x A(\alpha, x) \rightarrow \exists x \forall \alpha(\exists y \sigma(\overline{\alpha}(y)) > 0 \& \forall y(\sigma(\overline{\alpha}(y)) > 0 \rightarrow A(\alpha, \sigma(\overline{\alpha}(y) \cdot 1))))).
\]

$CC_{10}$ is a minor variation of, and is equivalent over $\textbf{IA}_1 + qf-AC_{00}$ to, Kleene and Vesley’s continuous choice schema *27.2 ("Brouwer’s Principle for numbers").

Proof. Assume $\beta \in 2^N$ and $\forall \alpha \exists x A^\beta(\alpha, x)$. By Lemma 2.5(b) it will be enough to find a $\sigma$ such that for all $\alpha:

(i) $\exists y(\sigma(\overline{\alpha}(y)) > 0 \lor Z(\beta))$ and

(ii) $\forall y(\sigma(\overline{\alpha}(y)) > 0 \lor Z(\gamma) \rightarrow A^\beta(\overline{\gamma}(\alpha, \sigma(\overline{\alpha}(y) \cdot 1))))$.

$CC_{10}$ provides a $\sigma$ such that for all $\alpha$:

(i’) $\exists y(\sigma(\overline{\alpha}(y)) > 0)$ and

(ii’) $\forall y(\sigma(\overline{\alpha}(y)) > 0 \rightarrow A^\beta(\alpha, \sigma(\overline{\alpha}(y) \cdot 1))))$.

Obviously (i’) entails (i). To prove (ii), let $y \in \mathbb{N}$ and $\gamma \in 2^N$. If $\sigma(\overline{\alpha}(y)) > 0$ then $A^\beta(\alpha, \sigma(\overline{\alpha}(y) \cdot 1))$ by (ii’) with Lemma 2.3, and if $Z(\gamma)$ then $A^\beta(\alpha, \sigma(\overline{\alpha}(y) \cdot 1))$ by Lemmas 2.4 and 2.3, so $\sigma(\overline{\alpha}(y)) > 0 \lor Z(\gamma) \rightarrow A^\beta(\alpha, \sigma(\overline{\alpha}(y) \cdot 1))$.

3.5. Lemma. $\textbf{IA}_1 + qf-AC_{00} + CC_{11} \vdash \forall \gamma \in 2^N(CC_{11})^\gamma$ where $CC_{11}$ is

\[
\forall \alpha \exists y A(\alpha, \beta) \rightarrow \exists y \forall \alpha \exists y[\forall x \exists y(\sigma((x + 1) * \overline{\alpha}(y)) = \beta(x) + 1 \& \forall z < y \sigma((x + 1) * \overline{\alpha}(z)) = 0) \& A(\alpha, \beta)],
\]

which is equivalent over $\textbf{IA}_1 + qf-AC_{00}$ to Kleene’s strongest continuous choice principle, “Brouwer’s Principle for functions” (axiom schema $x^27.1$ in [6]).
Corollary. 3.6. which suffices by the definition with Lemma 2.3.

Proof. Assume $\gamma \in 2^N$ and $\forall \alpha \exists \beta A^\gamma(\alpha, \beta)$. By Lemma 2.5(b) it will be enough to find a $\sigma$ such that

$$\forall \alpha \exists \beta \forall \delta \in 2^N | \exists x \exists y (\delta((x + 1) * \sigma(y))) = \beta(x) + 1$$

$$\land \forall z < y \sigma((x + 1) * \sigma(z)) = 0 \lor Z(\delta) \land A^\gamma(\alpha, \beta)].$$

CC$_{11}$ provides a $\sigma$ such that

$$\forall \alpha \exists \beta \forall x \exists y (\sigma((x + 1) * \sigma(y))) = \beta(x) + 1$$

$$\land \forall z < y \sigma((x + 1) * \sigma(z)) = 0 \land A^\gamma(\alpha, \beta)],$$

which suffices by the definition with Lemma 2.3.

3.6. Corollary. If $T$ is $IA_1$, Kleene’s neutral theory $B = IA_1 + AC_{001} + BI_1$, Kleene’s intuitionistic analysis $I = B + CC_{11}$ or any subsystem of $I$ obtained by adding to $IA_1$ any of the schemas $qf$-$AC_{00}$, $AC_{000}$, $AC_{01}$, $BI_1$ and/or $CC_{10}$, then $T + MP_1$ and $T$ prove the same $\Pi^0_2$ statements.

Proof. By Lemma 2.6(d), $T \vdash \forall \beta \in 2^N(MP_1)^\beta$. Hence by Theorem 3.1 with Lemmas 3.2 - 3.5, if $T + MP_1 \vdash E$ then $T \vdash \forall \beta \in 2^N E^\beta$.

If $E$ is $\forall \alpha \exists y A(x, y)$ where $A(x, y)$ has only bounded numerical quantifiers, then $A(x, y)$ is equivalent over $IA_1$ to a formula of the type described in Lemma 2.7(c), so by Theorem 3.1: if $T + MP_1 \vdash E$ then $T \vdash E^{\lambda x, 1}$ so $T \vdash E$.

3.7. Remarks. Lemma 2.7(c) holds also for formulas $E$ constructed from prime formulas and their negations using only $\land, \lor, \forall$ and $\exists$, in particular for all prenex formulas. It follows, for each subsystem $T$ of Kleene’s $I$ described in the statement of Corollary 3.6, that any prenex formula provable in $T + MP_1$ is provable in $T$.

Kleene’s original versions of the continuous choice principles would also satisfy Lemmas 3.4 and 3.5 over $IA_1 + qf$-$AC_{00}$. By Theorem 3.1 and Lemma 2.5, the equivalences between our versions and Kleene’s persist under the translation, and the proofs for $CC_{10}$ and $CC_{11}$ are simpler.

The question whether or not the “minimal” system $M = IA_1 + AC_{00}$! proves the same $\Pi^0_2$ formulas as $M + MP_1$ is still open, as far as we know, because $(\forall \exists \forall y A(x, y))^\alpha$ does not entail $\forall \exists x \exists y A^\alpha(x, y)$ unless $\alpha = \lambda x.1$. However, if

$$AC^\prime_{00} : \forall x(A(x) \lor B(x)) \rightarrow \exists x\forall x[(\alpha(x) = 0 \land A(x)) \lor (\alpha(x) \neq 0 \land B(x))]$$

is the axiom of countable choice for two alternatives, then $IRA + AC^\prime_{00} + MP_1$ is $\Pi^0_2$-conservative over $IRA + AC^\prime_{00}$ by Theorem 3.1. Since $AC^\prime_{00}$! is equivalent over $IRA$ to

$$\forall x(A(x) \lor \neg A(x)) \rightarrow \exists x\forall x(\alpha(x) = 0 \leftrightarrow A(x))$$

by [7], any prenex formula provable in $M + MP_1$ is provable in $IRA + AC^\prime_{00}$.

Because the translation $E \mapsto E^\beta$ essentially involves binary sequence quantifiers, it does not appear to solve the corresponding problem for $IA_1 + AC^R_{00}$ or for Solovay’s system $S = IA_1 + AC^R_{00} + BI_1$, where $AC^R_{00}$ is the restriction of $AC_{00}$ to arithmetical formulas $A(x, y)$ (with sequence parameters allowed). In the presence of bar induction, arithmetical countable choice interacts strongly with $MP_1$; e.g. Solovay showed that the classical version $S + (\neg \neg A \rightarrow A)$ of $S$ can be interpreted negatively in $IRA + BI_1 + DNS_1$. 2

2In fact, his proof justifies a stronger result: $S + (\neg \neg A \rightarrow A)$ can be interpreted negatively in $IRA + BI_1 + DNS_1$, where $DNS_1$ is the schema $\forall x(\neg \neg \exists x A(\overline{x}(x))) \rightarrow \neg \forall \exists x A(\overline{x}(x))$ for quantifier-free formulas $A(w)$. Another note with this and related results is in progress.
3.8. Acknowledgements. I am grateful to the editor for timely handling of this submission, and to both referees for their useful comments, questions and corrections. One referee suggested that the coding techniques in [1] might yield the \( \Pi^0_2 \)-conservativity of \( S + MP_1 \) over \( S \), but that is a project for younger minds. Any errors remaining are my own.

References