MARKOV'S PRINCIPLE AND SUBSYSTEMS OF INTUITIONISTIC ANALYSIS

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Abstract. Using a technique developed by Coquand and Hofmann [3] we verify that adding the analytical form MP_1: $\forall\alpha(\neg\neg\exists x\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0)$ of Markov's Principle does not increase the class of Π^0_2 formulas provable in Kleene and Vesley's formal system for intuitionistic analysis, or in subsystems obtained by omitting or restricting various axiom schemas in specified ways.

Introduction

In [6] Kleene proved that Markov's Principle MP_1 is neither provable nor refutable in his formal system **I** for intuitionistic analysis. By the Friedman-Dragalin translation, Markov's Rule is admissible for **I** and many subsystems.

We show that adding MP_1 as an axiom to **I** does not increase consistency strength, in the sense that no additional Π^0_2 formulas become provable. The method, adapted from Coquand and Hofmann's dynamic modification [3] of the Friedman-Dragalin translation, works also for subsystems of **I** with a few interesting exceptions.

1. Language, logic, and basic mathematical axioms

1.1. **The two-sorted formal language and intuitionistic predicate logic.** Kleene and Vesley's language $\mathcal{L}_1$ for two-sorted intuitionistic number theory or "intuitionistic analysis" has variables a,b,c,...,x,y,z,..., intended to range over natural numbers; variables $\alpha, \beta, \gamma, \ldots$, intended to range over one-place number-theoretic functions (choice sequences); finitely many constants $0, {}', +, \cdot, f_4, \ldots, f_p$, each representing a primitive recursive function or functional, where f_i has k_i places for number arguments and l_i places for type-1 function arguments; parentheses indicating function application; and Church's λ.

The *terms* (of type 0) and *functors* (of type 1) are defined inductively as follows. The number variables and 0 are terms. The function variables and each f_i with $k_i = 1, l_i = 0$ are functors. If $t_1, \ldots, t_{k_i}$ are terms and $u_1, \ldots, u_{l_i}$ are functors, then $f_i(t_1, \ldots, t_{k_i}, u_1, \ldots, u_{l_i})$ is a term. If x is a number variable and t is a term, then λx.t is a functor. And if u is a functor and t is a term, then (u)(t) is a term.

There is one relation symbol = for equality between terms; equality between functors u, v is defined extensionally by $u = v \equiv \forall x(u(x) = v(x))$ (where x is not free in u or v). The atomic formulas of $\mathcal{L}_1$ are the expressions s = t where s, t are terms. Composite formulas are defined inductively, using the connectives $\&, \vee, \rightarrow, \neg$, quantifiers $\forall, \exists$ of both sorts, and parentheses (often omitted under the usual conventions on scope). $A \leftrightarrow B$ is defined by $(A \rightarrow B) \,\&\, (B \rightarrow A)$.

The logical axioms and rules are those of two-sorted intuitionistic predicate logic, as presented in [6] (building on [4]). If the intuitionistic axiom schema

$\neg A \to (A \to B)$ were replaced by $\neg\neg A \to A$ (of which Markov's Principle MP_1 is a special case), two-sorted classical predicate logic would result.

1.2. **Two-sorted intuitionistic arithmetic $\mathbf{IA}_1$.** This is a conservative extension, in the language $\mathcal{L}_1$, of the first-order intuitionistic arithmetic $\mathbf{IA}_0$ in [4] based on $=, 0, ', +, \cdot$. The mathematical axioms of $\mathbf{IA}_1$ are:

(a) The axiom-schema of mathematical induction (for all formulas of $\mathcal{L}_1$): $A(0)\ \&\ \forall x(A(x) \to A(x')) \to A(x)$.
(b) The axioms of $\mathbf{IA}_0$ for $=, 0, ', +, \cdot$ (axioms 14-21 on page 82 of [4]) and the axioms expressing the primitive recursive definitions of the additional function constants $f_4, \ldots, f_{26}$ given in [6] and [5].[1]
(c) The open equality axiom: $x = y \to \alpha(x) = \alpha(y)$.
(d) The axiom-schema of λ-conversion: $(\lambda x.t(x))(s) = t(s)$, where $t(x)$ is a term and s is free for x in $t(x)$.

For readers familiar with [6], $\mathbf{IA}_1$ is the subsystem of the "basic system" **B** obtained by omitting the axiom schemas of countable choice and bar induction ($^{x}2.1$ and $^{x}26.3$, respectively).

In addition to the open equality axiom (c), the equality axioms

$$\alpha_1 = \beta_1\ \&\ldots\&\ \alpha_{l_i} = \beta_{l_i} \to f_i(x_1, \ldots, x_{k_i}, \alpha_1, \ldots, \alpha_{l_i}) = f_i(x_1, \ldots, x_{k_i}, \beta_1, \ldots, \beta_{l_i}),$$

are provable for all function constants f_i. Thus $\mathbf{IA}_1$ satisfies the replacement property of equality for functors as well as for terms.

$\mathbf{IA}_1$ can only prove the existence of primitive recursive sequences, in the sense that each closed theorem of the form $\exists\alpha A(\alpha)$ has a primitive recursive witness. The finite list of primitive recursive function constants, with their corresponding axioms, is intended to be expanded as needed. Here we use the λ notation to explicitly define termwise multiplication of sequences: $(\alpha \cdot \beta)$ will abbreviate $\lambda x(\alpha(x) \cdot \beta(x))$. We also define $sg(\alpha) = \lambda x.sg(\alpha(x))$, in effect adding binary sequence variables to $\mathcal{L}_1$.

1.3. **Intuitionistic recursive analysis IRA.** The principle of countable choice for numbers is expressed in $\mathcal{L}_1$ by the schema ($^*2.2$ in [6]):

$$\mathrm{AC}_{00}: \quad \forall x \exists y A(x, y) \to \exists\alpha\forall x A(x, \alpha(x)),$$

[1] $f_0 - f_3$ are $0, ', +, \cdot$ respectively. $f_4(a, b) = a^b$ (exponentiation), and $f_5, \ldots, f_{20}$ represent the primitive recursive function(al)s $a!$, $a \dot{-} b$, $pd(a)$, $min(a, b)$, $max(a, b)$, $\overline{sg}(a) = 1 \dot{-} a$, $sg(a) = 1 \dot{-} (1 \dot{-} a)$, $|a - b|$, $rm(a, b)$, $[a/b]$, $\Sigma_{y<b}\alpha(y)$, $\Pi_{y<b}\alpha(y)$, $min_{y\leq b}\alpha(y)$, $max_{y\leq b}\alpha(y)$, p_a (the a^{th} prime, with $p_0 = 2$), and $(a)_i$ (the exponent of p_i in the prime factorization of a) respectively. We write $(a)_i$ for $f_{20}(a, i)$, and similarly for the other function constants. $f_{21}(a) = lh(a) = \Sigma_{i<a} sg((a)_i)$ represents the number of positive exponents in the prime factorization of a. Bounded quantifiers are defined with the help of bounded sum and product. Seq(a) is a prime formula equivalent to $a > 0\ \&\ \forall i < lh(a)\ (a)_i > 0$, expressing "$a$ codes the finite sequence $((a)_0 - 1, \ldots, (a)_{lh(a)-1} - 1)$". $f_{22}(a, b) = a * b$ produces a code for the concatenation of two finite sequences from their codes. $\langle\,\rangle = 1$ codes the empty sequence, and $f_{23}(x, \alpha) = \overline{\alpha}(x) = \Pi_{i<x} p_i^{\alpha(i)+1}$ represents the standard code $\langle \alpha(0) + 1, \ldots, \alpha(x - 1) + 1\rangle$ for the x^{th} initial segment of α. This coding is not onto $\mathbb{N}$, but it satisfies $\langle a_0 + 1, \ldots, a_k + 1\rangle * \langle a_{k+1} + 1, \ldots, a_m + 1\rangle = \langle a_0 + 1, \ldots, a_m + 1\rangle$. In contrast, $f_{24}(x, \alpha) = \tilde{\alpha}(x) = \Pi_{i<x} p_i^{\alpha(i)}$ cannot code finite sequences directly as $\langle a_0, \ldots, a_k\rangle = \langle a_0, \ldots, a_k, 0\rangle$. $f_{25}(a, b) = a \circ b = \Pi_{i<max(a,b)} p_i^{max((a)_i,(b)_i)}$, and $f_{26}(y) = ccp(y)$ represents the course-of-values function for the characteristic function of the predicate "y is a computation tree number." These suffice for Kleene's formal treatment ([5] Part I) of recursive partial functionals, including the recursion theorem and a normal form theorem.

where α, x must be free for y in $\mathrm{A(x, y)}$. Intuitionistic recursive analysis **IRA** can be axiomatized, as a subsystem of Kleene and Vesley's **B**, by $\mathbf{IA}_1$ + qf-AC_{00}, where qf-AC_{00} is the restriction of AC_{00} to formulas $\mathrm{A(x, y)}$ without sequence quantifiers and with only bounded number quantifiers. **IRA** ensures that the range of the type-1 variables contains all general recursive sequences and is closed under general recursive processes. Troelstra's **EL** and Veldman's **BIM** are alternative axiomatizations of **IRA**, cf. [8], [7].

In the two-sorted language, **IRA** + MP_1 + CT_1 formalizes Russian recursive analysis (**RUSS** in [2]), where MP_1 is the functional form of Markov's Principle

$$\mathrm{MP}_1: \quad \forall\alpha[\neg\neg\exists \mathrm{x}\alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\alpha(\mathrm{x}) = 0]$$

and CT_1 expresses Church's Thesis:

$$\mathrm{CT}_1: \quad \forall\alpha\exists \mathrm{e}\forall \mathrm{x}\exists \mathrm{y}[\mathrm{T}_0(\mathrm{e, x, y}) \ \& \ \mathrm{U(y)} = \alpha(\mathrm{x})].$$

The general recursive functions form a classical ω-model of **RUSS** and hence of **IRA**, but **RUSS** + AC_{00} (unlike **IRA** + AC_{00}) is inconsistent with classical logic.

2. Definition of the Translation, and Properties Proved in $\mathbf{IA}_1$

2.1. **Definition.** Let $\mathrm{Z}(\alpha)$ abbreviate $\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$. To each formula E of $\mathcal{L}_1$ and each sequence variable α not occurring in E, we associate another formula E^α with the same free variables plus α, by induction on the logical form of E as follows. For cases 4 and 5, β should be distinct from α, and $\mathrm{A}^{\mathrm{sg}(\beta)}$ is the result of substituting $\mathrm{sg}(\beta)$ for γ in the definition of A^γ. Similarly for $\mathrm{B}^{\alpha\cdot\beta}$ in Case 4.

(1) P^α is $\mathrm{P} \vee \mathrm{Z}(\alpha)$ if P is prime.
(2) $(\mathrm{A} \ \& \ \mathrm{B})^\alpha$ is $\mathrm{A}^\alpha \ \& \ \mathrm{B}^\alpha$.
(3) $(\mathrm{A} \vee \mathrm{B})^\alpha$ is $\mathrm{A}^\alpha \vee \mathrm{B}^\alpha$.
(4) $(\mathrm{A} \rightarrow \mathrm{B})^\alpha$ is $\forall\beta(\mathrm{A}^{\mathrm{sg}(\beta)} \rightarrow \mathrm{B}^{\alpha\cdot\beta})$.
(5) $(\neg \mathrm{A})^\alpha$ is $\forall\beta(\mathrm{A}^{\mathrm{sg}(\beta)} \rightarrow \mathrm{Z}(\alpha\cdot\beta))$.
(6) $(\forall \mathrm{x}\mathrm{A}(\mathrm{x}))^\alpha$ is $\forall \mathrm{x}\mathrm{A}^\alpha(\mathrm{x})$.
(7) $(\exists \mathrm{x}\mathrm{A}(\mathrm{x}))^\alpha$ is $\exists \mathrm{x}\mathrm{A}^\alpha(\mathrm{x})$.
(8) $(\forall\gamma\mathrm{A}(\gamma))^\alpha$ is $\forall\gamma\mathrm{A}^\alpha(\gamma)$.
(9) $(\exists\gamma\mathrm{A}(\gamma))^\alpha$ is $\exists\gamma\mathrm{A}^\alpha(\gamma)$.

From now on, let $\alpha \in 2^{\mathbb{N}}$ abbreviate $\alpha = \mathrm{sg}(\alpha)$.

2.2. **Proposition.**

(a) $\mathbf{IA}_1 \vdash \forall\alpha\forall\beta(\mathrm{Z}(\alpha\cdot\beta) \leftrightarrow \mathrm{Z}(\alpha) \vee \mathrm{Z}(\beta))$.
(b) $\mathbf{IA}_1 \vdash \forall\alpha(\mathrm{E}^\alpha \leftrightarrow \mathrm{E}^{(\mathrm{sg}(\alpha))})$ for all formulas E.
(c) $\mathbf{IA}_1 \vdash \forall\alpha \in 2^{\mathbb{N}}(\mathrm{E}(\alpha) \leftrightarrow \mathrm{E}(\mathrm{sg}(\alpha)))$.

Proofs. (a) holds by intuitionistic logic, (b) is proved by formula induction, and the replacement property of equality for functors guarantees (c).

2.3. **Lemma.** $\mathbf{IA}_1 \vdash \forall\alpha\forall\beta\forall\gamma(\mathrm{E}^\alpha \ \& \ \gamma = \alpha\cdot\beta \rightarrow \mathrm{E}^\gamma)$.

Proof. Only Cases 4 and 5 require attention. If E is $\mathrm{A} \rightarrow \mathrm{B}$ where A, B both satisfy the lemma, assume $(\mathrm{A} \rightarrow \mathrm{B})^\alpha \ \& \ \gamma = \alpha\cdot\beta$. If $\mathrm{A}^{\mathrm{sg}(\delta)}$ then $\mathrm{B}^{\alpha\cdot\delta}$ by definition of $(\mathrm{A} \rightarrow \mathrm{B})^\alpha$, and $\delta\cdot\gamma = (\alpha\cdot\delta)\cdot\beta$ so $\mathrm{B}^{\delta\cdot\gamma}$ by the induction hypothesis on B. So $(\mathrm{A} \rightarrow \mathrm{B})^\gamma$.

If E is $\neg\mathrm{A}$ where A satisfies the lemma, assume $(\neg\mathrm{A})^\alpha \ \& \ \gamma = \alpha\cdot\beta$. If $\mathrm{A}^{\mathrm{sg}(\delta)}$, then $\mathrm{Z}(\alpha\cdot\delta)$ by definition of $(\neg\mathrm{A})^\alpha$, so $\mathrm{Z}(\gamma\cdot\delta)$ by Proposition 2.2(a). So $(\neg\mathrm{A})^\gamma$.

2.4. **Lemma.** $\mathbf{IA}_1 \vdash \forall\alpha(\mathrm{Z}(\alpha) \rightarrow \mathrm{E}^\alpha)$ for all formulas E.

2.5. **Lemma.**

(a) $\mathbf{IA}_1 \vdash \forall\alpha \in 2^{\mathbb{N}}((\mathrm{A} \to \mathrm{B})^\alpha \to (\mathrm{A}^\alpha \to \mathrm{B}^\alpha))$.
(b) $\mathbf{IA}_1 \vdash \forall\alpha \in 2^{\mathbb{N}}(\mathrm{A} \to \mathrm{B})^\alpha \leftrightarrow \forall\alpha \in 2^{\mathbb{N}}(\mathrm{A}^\alpha \to \mathrm{B}^\alpha)$.

Proofs. (a) follows immediately from the definition and Proposition 2.2(b) with the fact that $\alpha \cdot \alpha = \alpha$ for all $\alpha \in 2^{\mathbb{N}}$.

For (b), the implication from left to right follows from (a) by logic. For the converse assume $\forall\alpha \in 2^{\mathbb{N}}(\mathrm{A}^\alpha \to \mathrm{B}^\alpha)$ and $\alpha \in 2^{\mathbb{N}}$ and $\mathrm{A}^{\mathrm{sg}(\beta)}$; then $\mathrm{B}^{\mathrm{sg}(\beta)}$ by the assumption, so B^β by Proposition 2.2(b), so $\mathrm{B}^{\alpha\cdot\beta}$ by Lemma 2.3. So $(\mathrm{A} \to \mathrm{B})^\alpha$.

2.6. **Lemma.** If E is $\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$ (i.e., $\mathrm{Z}(\alpha)$) then $\mathbf{IA}_1$ proves:

(a) $\forall\beta \in 2^{\mathbb{N}}(\mathrm{E}^\beta \leftrightarrow \mathrm{E} \vee \mathrm{Z}(\beta))$.
(b) $\forall\beta \in 2^{\mathbb{N}}((\neg\mathrm{E})^\beta \leftrightarrow (\mathrm{E} \to \mathrm{Z}(\beta)))$.
(c) $\forall\beta \in 2^{\mathbb{N}}((\neg\neg\mathrm{E})^\beta \leftrightarrow \mathrm{E} \vee \mathrm{Z}(\beta))$.
(d) $\forall\beta \in 2^{\mathbb{N}}(\neg\neg\mathrm{E} \leftrightarrow \mathrm{E})^\beta$.

Proofs. (a) is immediate by Definition 2.1 with intuitionistic logic. For (b), under the assumption $\beta \in 2^{\mathbb{N}}$ and using (a), Proposition 2.2, intuitionistic logic and the fact that $\beta \cdot \beta = \beta$, we have the following chain of equivalences:

$$\begin{aligned}(\neg\mathrm{E})^\beta &\leftrightarrow \forall\gamma \in 2^{\mathbb{N}}(\mathrm{E}^\gamma \to \mathrm{Z}(\beta\cdot\gamma))\\ &\leftrightarrow \forall\gamma \in 2^{\mathbb{N}}(\mathrm{E} \vee \mathrm{Z}(\gamma) \to \mathrm{Z}(\beta\cdot\gamma))\\ &\leftrightarrow \forall\gamma \in 2^{\mathbb{N}}(\mathrm{E} \to \mathrm{Z}(\beta\cdot\gamma)) \leftrightarrow (\mathrm{E} \to \mathrm{Z}(\beta)).\end{aligned}$$

For (c), under the assumption $\beta \in 2^{\mathbb{N}}$, by (b) we have

$$(\neg\neg\mathrm{E})^\beta \leftrightarrow \forall\gamma \in 2^{\mathbb{N}}((\neg\mathrm{E})^\gamma \to \mathrm{Z}(\beta\cdot\gamma)) \leftrightarrow \forall\gamma \in 2^{\mathbb{N}}((\mathrm{E} \to \mathrm{Z}(\gamma)) \to \mathrm{Z}(\beta\cdot\gamma)).$$

If $\forall\gamma \in 2^{\mathbb{N}}((\mathrm{E} \to \mathrm{Z}(\gamma)) \to \mathrm{Z}(\beta\cdot\gamma))$, let $\gamma = \mathrm{sg}(\alpha)$; then $\mathrm{Z}(\gamma) \leftrightarrow \mathrm{Z}(\alpha)$ and $\gamma \in 2^{\mathbb{N}}$. Then $(\mathrm{Z}(\gamma) \to \mathrm{Z}(\gamma)) \to \mathrm{Z}(\beta\cdot\gamma)$ since E is $\mathrm{Z}(\alpha)$, so $\mathrm{Z}(\beta\cdot\gamma)$, so $\mathrm{Z}(\beta) \vee \mathrm{Z}(\gamma)$ by Proposition 2.2(a), so $\mathrm{Z}(\beta) \vee \mathrm{E}$, so $\mathrm{E} \vee \mathrm{Z}(\beta)$. For the converse use Proposition 2.2(a). Then (d) follows from (a) and (c) with Lemma 2.5(b).

2.7. **Lemma.**

(a) If E has no $\to$ or $\neg$ then $\mathbf{IA}_1 \vdash (\mathrm{E}^{\lambda \mathrm{z}.1} \leftrightarrow \mathrm{E})$.
(b) $\mathbf{IA}_1 \vdash (\neg\mathrm{A})^{\lambda \mathrm{z}.1} \to \neg(\mathrm{A}^{\lambda \mathrm{z}.1})$ for all formulas A.
(c) If E is constructed from prime formulas and their negations using only & and $\vee$, then $\mathbf{IA}_1 \vdash (\mathrm{E}^{\lambda \mathrm{z}.1} \to \mathrm{E})$.

3. Applications to Subsystems of Kleene's Formal System **I** for Intuitionistic Analysis

3.1. **Theorem.** If $\mathbf{T}$ is a theory extending $\mathbf{IA}_1$ by axioms and axiom schemas $\mathrm{F}_1, \dots, \mathrm{F}_n$ such that $\mathbf{T} \vdash \forall\beta \in 2^{\mathbb{N}}\, (\mathrm{F}_i)^\beta$ for $i = 1, \dots, n$, and if E is derivable in $\mathbf{T}$ from assumptions $\mathrm{A}_1, \dots, \mathrm{A}_m$ with all free variables held constant in the deduction, then E^β is derivable in $\mathbf{T}$ from the assumptions $\beta \in 2^{\mathbb{N}}$, $(\mathrm{A}_1)^\beta, \dots, (\mathrm{A}_m)^\beta$ with all free variables held constant.

Proof. $\mathbf{IA}_1 \vdash \forall\alpha \in 2^{\mathbb{N}}\, \mathrm{E}^\alpha$ when E is any axiom of $\mathbf{IA}_1$, using the lemmas in the previous section with $\forall\alpha \in 2^{\mathbb{N}}(\alpha\cdot\alpha = \alpha)$ as appropriate (e.g. for the mathematical induction schema). If B^β and $(\mathrm{B} \to \mathrm{C})^\beta$ are derivable in $\mathbf{IA}_1$ from $\beta = \mathrm{sg}(\beta)$, $(\mathrm{A}_1)^\beta, \dots, (\mathrm{A}_m)^\beta$ with the free variables held constant, then by Lemma 2.5(a) so is $\mathrm{B}^\beta \to \mathrm{C}^\beta$, and therefore also C^β. Similarly for the other rules of inference.

3.2. **Lemma.** $\mathbf{IA}_1$ + $\mathrm{AC}_{00} \vdash \forall\beta \in 2^{\mathbb{N}}(\mathrm{AC}_{00})^{\beta}$, and similarly for qf-$\mathrm{AC}_{00}$, AC_{01}.

Proofs. By the definition with Lemma 2.5(b).

3.3. **Lemma.** $\mathbf{IA}_1$ + $\mathrm{BI}_1 \vdash \forall\beta \in 2^{\mathbb{N}}(\mathrm{BI}_1)^{\beta}$ where BI_1 is the bar induction schema

$$\forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0 \;\&\; \forall w(\mathrm{Seq}(w) \;\&\; \rho(w) = 0 \to A(w)) \\ \&\; \forall w(\mathrm{Seq}(w) \;\&\; \forall s A(w * \langle s+1\rangle) \to A(w)) \to A(\langle\,\rangle).$$

Proof. Assume $\beta \in 2^{\mathbb{N}}$ and

(i) $(\forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0)^{\beta}$,
(ii) $(\forall w(\mathrm{Seq}(w) \;\&\; \rho(w) = 0 \to A(w)))^{\beta}$,
(iii) $(\forall w(\mathrm{Seq}(w) \;\&\; \forall s A(w * \langle s+1\rangle) \to A(w)))^{\beta}$.

By Lemma 2.5 it will be enough to prove $A^{\beta}(\langle\,\rangle)$. By the definition and the lemmas in the previous section, over $\mathbf{IA}_1$ the numbered assumptions are equivalent respectively to

(i') $\forall\alpha\exists x(\rho(\overline{\alpha}(x)) = 0 \vee Z(\beta))$,
(ii') $\forall w\forall\gamma \in 2^{\mathbb{N}}((\mathrm{Seq}(w) \;\&\; \rho(w) = 0) \vee Z(\gamma) \to A^{\beta\cdot\gamma}(w)))$,
(iii') $\forall w\forall\gamma \in 2^{\mathbb{N}}(\mathrm{Seq}(w) \;\&\; \forall s A^{\gamma}(w * \langle s+1\rangle) \to A^{\beta\cdot\gamma}(w))$.

In $\mathbf{IA}_1$ we may define $\sigma \in 2^{\mathbb{N}}$ so that

$$\sigma(w) = 0 \leftrightarrow \rho(w) = 0 \vee \exists x \leq w\beta(x) = 0.$$

From (i') it follows immediately that $\forall\alpha\exists x\sigma(\overline{\alpha}(x) = 0)$. From (ii') with $\gamma = \beta$ and the fact that $\beta = \beta \cdot \beta$ we have $\forall w(\mathrm{Seq}(w) \;\&\; \sigma(w) = 0 \to A^{\beta}(w))$. From (iii') similarly, $\forall w(\mathrm{Seq}(w) \;\&\; \forall s A^{\beta}(w * \langle s+1\rangle) \to A^{\beta}(w))$, so $A^{\beta}(\langle\,\rangle)$ follows by BI_1.

3.4. **Lemma.** $\mathbf{IA}_1$ + $\mathrm{CC}_{10} \vdash \forall\beta \in 2^{\mathbb{N}}\,(\mathrm{CC}_{10})^{\beta}$ where CC_{10} is

$$\forall\alpha\exists x A(\alpha, x) \to \exists\sigma\forall\alpha(\exists y\sigma(\overline{\alpha}(y)) > 0 \;\&\; \forall y(\sigma(\overline{\alpha}(y)) > 0 \to A(\alpha, \sigma(\overline{\alpha}(y)\dot{-}1)))).$$

CC_{10} is a minor variation of, and is equivalent over $\mathbf{IA}_1$ + qf-AC_{00} to, Kleene and Vesley's continuous choice schema *27.2 ("Brouwer's Principle for numbers").

Proof. Assume $\beta \in 2^{\mathbb{N}}$ and $\forall\alpha\exists x A^{\beta}(\alpha, x)$. By Lemma 2.5(b) it will be enough to find a σ such that for all α:

(i) $\exists y(\sigma(\overline{\alpha}(y)) > 0 \vee Z(\beta))$ and
(ii) $\forall y\forall\gamma \in 2^{\mathbb{N}}(\sigma(\overline{\alpha}(y)) > 0 \vee Z(\gamma) \to A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)\dot{-}1)))$.

CC_{10} provides a σ such that for all α:

(i') $\exists y\; \sigma(\overline{\alpha}(y)) > 0$ and
(ii') $\forall y(\sigma(\overline{\alpha}(y)) > 0 \to A^{\beta}(\alpha, \sigma(\overline{\alpha}(y)\dot{-}1)))$.

Obviously (i') entails (i). To prove (ii), let $y \in \mathbb{N}$ and $\gamma \in 2^{\mathbb{N}}$. If $\sigma(\overline{\alpha}(y)) > 0$ then $A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)\dot{-}1))$ by (ii') with Lemma 2.3, and if $Z(\gamma)$ then $A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)\dot{-}1))$ by Lemmas 2.4 and 2.3, so $\sigma(\overline{\alpha}(y)) > 0 \vee Z(\gamma) \to A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)\dot{-}1))$.

3.5. **Lemma.** $\mathbf{IA}_1$ + qf-AC_{00} + $\mathrm{CC}_{11} \vdash \forall\gamma \in 2^{\mathbb{N}}\,(\mathrm{CC}_{11})^{\gamma}$ where CC_{11} is

$$\forall\alpha\exists\beta A(\alpha, \beta) \to \exists\sigma\forall\alpha\exists\beta[\forall x\exists y(\sigma(\langle x+1\rangle * \overline{\alpha}(y)) = \beta(x) + 1 \\ \&\; \forall z < y\,\sigma(\langle x+1\rangle * \overline{\alpha}(z)) = 0) \;\&\; A(\alpha, \beta)],$$

which is equivalent over $\mathbf{IA}_1$ + qf-AC_{00} to Kleene's strongest continuous choice principle, "Brouwer's Principle for functions" (axiom schema $^{x}27.1$ in [6]).

Proof. Assume $\gamma \in 2^{\mathbb{N}}$ and $\forall\alpha\exists\beta \mathrm{A}^{\gamma}(\alpha,\beta)$. By Lemma 2.5(b) it will be enough to find a σ such that

$$\forall\alpha\exists\beta\forall\delta \in 2^{\mathbb{N}}[\forall \mathrm{x}\exists \mathrm{y}((\sigma(\langle \mathrm{x}+1\rangle * \overline{\alpha}(\mathrm{y})) = \beta(\mathrm{x})+1 \\ \& \ \forall \mathrm{z} < \mathrm{y}\, \sigma(\langle \mathrm{x}+1\rangle * \overline{\alpha}(\mathrm{z})) = 0) \vee \mathrm{Z}(\delta)) \ \& \ \mathrm{A}^{\gamma\cdot\delta}(\alpha,\beta)].$$

CC_{11} provides a σ such that

$$\forall\alpha\exists\beta[\forall \mathrm{x}\exists \mathrm{y}(\sigma(\langle \mathrm{x}+1\rangle * \overline{\alpha}(\mathrm{y})) = \beta(\mathrm{x})+1 \\ \& \ \forall \mathrm{z} < \mathrm{y}\, \sigma(\langle \mathrm{x}+1\rangle * \overline{\alpha}(\mathrm{z})) = 0) \ \& \ \mathrm{A}^{\gamma}(\alpha,\beta)],$$

which suffices by the definition with Lemma 2.3.

3.6. **Corollary.** If $\mathbf{T}$ is $\mathbf{IA}_1$, Kleene's neutral theory $\mathbf{B} = \mathbf{IA}_1 + \mathrm{AC}_{01} + \mathrm{BI}_1$, Kleene's intuitionistic analysis $\mathbf{I} = \mathbf{B} + \mathrm{CC}_{11}$ or any subsystem of $\mathbf{I}$ obtained by adding to $\mathbf{IA}_1$ any of the schemas qf-AC_{00}, AC_{00}, AC_{01}, BI_1 and/or CC_{10}, then $\mathbf{T} + \mathrm{MP}_1$ and $\mathbf{T}$ prove the same Π^0_2 statements.

Proof. By Lemma 2.6(d), $\mathbf{T} \vdash \forall\beta \in 2^{\mathbb{N}}(\mathrm{MP}_1)^{\beta}$. Hence by Theorem 3.1 with Lemmas 3.2 - 3.5, if $\mathbf{T} + \mathrm{MP}_1 \vdash \mathrm{E}$ then $\mathbf{T} \vdash \forall\beta \in 2^{\mathbb{N}}\ \mathrm{E}^{\beta}$.

If E is $\forall \mathrm{x}\exists \mathrm{y}\mathrm{A}(\mathrm{x},\mathrm{y})$ where $\mathrm{A}(\mathrm{x},\mathrm{y})$ has only bounded numerical quantifiers, then $\mathrm{A}(\mathrm{x},\mathrm{y})$ is equivalent over $\mathbf{IA}_1$ to a formula of the type described in Lemma 2.7(c), so by Theorem 3.1: if $\mathbf{T} + \mathrm{MP}_1 \vdash \mathrm{E}$ then $\mathbf{T} \vdash \mathrm{E}^{\lambda \mathrm{z}.1}$ so $\mathbf{T} \vdash \mathrm{E}$.

3.7. **Remarks.** Lemma 2.7(c) holds also for formulas E constructed from prime formulas and their negations using only $\&, \vee, \forall$ and $\exists$, in particular for all prenex formulas. It follows, for each subsystem $\mathbf{T}$ of Kleene's $\mathbf{I}$ described in the statement of Corollary 3.6, that any prenex formula provable in $\mathbf{T} + \mathrm{MP}_1$ is provable in $\mathbf{T}$.

Kleene's original versions of the continuous choice principles would also satisfy Lemmas 3.4 and 3.5 over $\mathbf{IA}_1$ + qf-AC_{00}. By Theorem 3.1 and Lemma 2.5, the equivalences between our versions and Kleene's persist under the translation, and the proofs for CC_{10} and CC_{11} are simpler.

The question whether or not the "minimal" system $\mathbf{M} = \mathbf{IA}_1 + \mathrm{AC}_{00}!$ proves the same Π^0_2 formulas as $\mathbf{M} + \mathrm{MP}_1$ is still open, as far as we know, because $(\forall \mathrm{x}\exists! \mathrm{y}\mathrm{A}(\mathrm{x},\mathrm{y}))^{\alpha}$ does not entail $\forall \mathrm{x}\exists! \mathrm{y}\mathrm{A}^{\alpha}(\mathrm{x},\mathrm{y})$ unless $\alpha = \lambda \mathrm{x}.1$. However, if

$$\mathrm{AC}^{\vee}_{00}: \quad \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \mathrm{B}(\mathrm{x})) \rightarrow \exists\alpha\forall \mathrm{x}[(\alpha(\mathrm{x}) = 0 \ \& \ \mathrm{A}(\mathrm{x})) \vee (\alpha(\mathrm{x}) \neq 0 \ \& \ \mathrm{B}(\mathrm{x}))]$$

is the axiom of countable choice for two alternatives, then $\mathbf{IRA} + \mathrm{AC}^{\vee}_{00} + \mathrm{MP}_1$ is Π^0_2-conservative over $\mathbf{IRA} + \mathrm{AC}^{\vee}_{00}$ by Theorem 3.1. Since $\mathrm{AC}_{00}!$ is equivalent over $\mathbf{IRA}$ to

$$\forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \neg\mathrm{A}(\mathrm{x})) \rightarrow \exists\alpha\forall \mathrm{x}(\alpha(\mathrm{x}) = 0 \leftrightarrow \mathrm{A}(\mathrm{x}))$$

by [7], any prenex formula provable in $\mathbf{M} + \mathrm{MP}_1$ is provable in $\mathbf{IRA} + \mathrm{AC}^{\vee}_{00}$.

Because the translation $\mathrm{E} \mapsto \mathrm{E}^{\beta}$ essentially involves binary sequence quantifiers, it does not appear to solve the corresponding problem for $\mathbf{IA}_1 + \mathrm{AC}^{Ar}_{00}$ or for Solovay's system $\mathbf{S} = \mathbf{IA}_1 + \mathrm{AC}^{Ar}_{00} + \mathrm{BI}_1$, where AC^{Ar}_{00} is the restriction of AC_{00} to arithmetical formulas $\mathrm{A}(\mathrm{x},\mathrm{y})$ (with sequence parameters allowed). In the presence of bar induction, arithmetical countable choice interacts strongly with MP_1; e.g. Solovay showed that the classical version $\mathbf{S} + (\neg\neg\mathrm{A} \rightarrow \mathrm{A})$ of $\mathbf{S}$ can be interpreted negatively in $\mathbf{IRA} + \mathrm{BI}_1 + \mathrm{MP}_1$.[2]

[2]In fact, his proof justifies a stronger result: $\mathbf{S} + (\neg\neg\mathrm{A} \rightarrow \mathrm{A})$ can be interpreted negatively in $\mathbf{IRA} + \mathrm{BI}_1 + \mathrm{DNS}_1$, where DNS_1 is the schema $\forall\alpha\neg\neg\exists \mathrm{x}\mathrm{A}(\overline{\alpha}(\mathrm{x})) \rightarrow \neg\neg\forall\alpha\exists \mathrm{x}\mathrm{A}(\overline{\alpha}(\mathrm{x}))$ for quantifier-free formulas $\mathrm{A}(\mathrm{w})$. Another note with this and related results is in progress.

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