AIM WORKSHOP: HIGHER-DIMENSIONAL LOG CALABI–YAU PAIRS

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Contents

Throughout this note, we work over the field $\mathbb{C}$ of complex numbers. All algebraic varieties are assumed to be normal and quasi-projective unless otherwise stated.

1. Toric varieties

1.1. **Toric varieties via torus actions.** A n-dimensional projective variety X is said to be *toric* if it admits an effective action of a n-dimensional algebraic torus $\mathbb{G}_m^n$. Toric varieties can be described combinatorially and hence they form a class of algebraic varieties in which computations can be performed quite explicitly. Weakening the previous definition leads to the definition of $\mathbb{T}$-*varieties*. A projective $\mathbb{T}$-*variety of torus complexity* k is a normal projective variety X of dimension n endowed with the action of an algebraic torus $\mathbb{G}_m^{n-k}$.[1] Complexity one $\mathbb{T}$-varieties have been extensively studied [1, 8]. Although they have a mostly combinatorial description through the language of *divisorial fans*, their behavior is quite far from toric varieties. They may not be rational and they may not have rational singularities, Du Bois singularities, nor log canonical singularities [12, 13]. Anyhow, the development of complexity one $\mathbb{T}$-varieties has been important in Algebraic Geometry (see, e.g., [10, 11]).

We turn to consider two characterizations of toric varieties. Each of these definitions will lead to new classes of algebraic varieties when considering a *weakened* version of the definition. In this note, we will focus on three definitions:

(1) Complexity one varieties,
(2) Log rational varieties, and
(3) Cluster type varieties.

The first class has been mentioned in [3]. However, it has not been studied extensively. The latter classes have been defined in [5] and their understanding is still embryonic. Then, we proceed to introduce several questions and open problems about these classes of varieties and related invariants.

[1] In the literature, these varieties are known as $\mathbb{T}$ *varieties of complexity* k or *complexity* k $\mathbb{T}$-*varieties*. However, this name conflicts with some definitions below. Furthermore, adding the word "torus" emphasizes in the existence of a torus action.

1.2. **Toric varieties via the complexity.** First, we focus on the characterization of toric varieties in terms of their complexity. The *complexity* of a log Calabi–Yau pair (X, B) is the following value

$$c(X, B) := \dim X + \dim \mathrm{Cl}_{\mathbb{Q}}(X) - |B|,$$

where $|B|$ stands for the sum of the coefficients of B. The following is a theorem due to Brown, M$^{\mathrm{c}}$Kernan, Svaldi, and Zhong (see [3]).

Theorem 1.1. *Let (X, B) be a log Calabi–Yau pair. Then, the inequality $c(X, B) \geq 0$ holds. Furthermore, if $c(X, B) < 1$, then the pair $(X, \lfloor B \rfloor)$ is toric*[2]

The previous theorem is known as the characterization of toric varieties via the complexity. We may define the *absolute complexity* of a normal projective variety X to be

$$c(X) := \inf\{c(X, B) \mid \ (X, B) \text{ is log Calabi–Yau}\}.$$

We set the absolute complexity to be infinite if the set in the definition is empty. We may drop the word *absolute* whenever it is clear from the context, for instance, when X is not a Calabi–Yau variety. In [3, §7], the authors give an example of a normal projective variety of complexity one which is not rational. This implies that the inequality in Theorem 1.1 is sharp. It is unclear what values are complexities of algebraic varieties. This is our first question.

Question 1.2. *Does the set of absolute complexities of all n-dimensional varieties satisfy the ascending chain condition? If so, is it discrete?*

The previous question is motivated by Birkar's boundedness of complements [2]. A positive answer would follow if the complexity is computed by a bounded complement, i.e., an element of $|-mK_X|$ where the integer m is controlled by the dimension of X. We know that algebraic varieties of complexity zero are toric. Then, it is natural to study more extensively the algebraic varieties of complexity one. For instance, the following question is natural from the perspective of the theory of complements:

Question 1.3. *Let X be a normal projective variety of complexity one. Does there exists a reduced boundary B on X for which (X, B) is a log Calabi–Yau paur of index two* [3] *and coregularity zero?*

The coregularity of a pair is the dimension of its smallest log canonical center on a dlt modificaton (see, e.g., [6, 15]). The previous question is motivated by the fact that the spectrum of the Cox ring of a Mori dream space of complexity one has only compound A_n singularities. Compound A_n singularities admit 1-complements of coregularity zero. Varieties of complexity one can be obtained by blowing up toric varieties along non-toric strata. In this case, one can show the existence of a 1-complement of coregularity zero. However, as mentioned above, varieties of complexity one may not be rational. This motivates the following question.

Question 1.4. *Can we detect the rationality of a complexity one variety via the theory of complements?*

In plain words, the previous question asks whether we can understand the rationality of X by finding elements in $|-mK_X|$ with special properties. In [3, §7], the authors mention that a non-rational complexity one variety has 2-torsion on its class group. However, the blow-up of a toric variety (which is rational), may also have 2-torsion on its class group. Thus, the (non)-existence of torsion in the class group does not give a criteria for rationality. On the other hand, Mauri and Filipazzi have related the orientability of dual complex to the index of Calabi–Yau pairs [6]. As a first attempt to solve Question 1.4, one may ask whether a rational complexity one variety admits a 1-complement of coregularity zero.

[2]This means that X is a toric variety and $\lfloor B \rfloor$ is a torus invariant divisor.

[3]This means that $2(K_X + B) \sim 0$ and in this case B is called a 2-*complement*.

1.3. **Toric varieties via birational maps.** In this subsection, we give a characterization of toric pairs in terms of birational geometry. Throughout the rest of this note, we write Σ^n for the reduced sum of the hyperplane coordinates of $\mathbb{P}^n$. We say that two log pairs (X, B) and (Y, Δ) are *crepant birational equivalent*, denoted by $(X, B) \simeq_{\rm cbir} (Y, \Delta)$ if the following conditions are satisfied:

(1) there is a common resolution $p\colon Z \to X$ and $q\colon Z \to Y$, and
(2) we have $p^*(K_X + B) = q^*(K_Y + \Delta)$.

In the previous context, the pair (X, B) is log Calabi–Yau if and only if (Y, Δ) is so. Furthermore, many invariants of the log Calabi–Yau pairs, such as the index, coregularity, and dual complex are preserved under crepant birational transformations.

Let (X, B) be a log Calabi–Yau pair. The pair (X, B) is *toric* if X is a toric variety and B is the reduced sum of all the prime torus invariant divisors. In this case, we say that (X, B) is a toric log Calabi–Yau pair. The following is a characterization of toric pairs via birational maps (see, e.g., [5, Theorem 1.11]).

Theorem 1.5. *Let (X, B) be a log Calabi–Yau pair. Then, the pair (X, B) is toric if and only if there exists a crepant birational map $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ that induces a small birational map $\mathbb{G}_m^n \dashrightarrow X \setminus B$.*

In other words, roughly speaking, a pair (X, B) is toric if and only if it can be transformed into $(\mathbb{P}^n, \Sigma^n)$ by blowing up strata of B and contracting components of B. Weakening the characterization given by Theorem 1.5 naturally leads to the following definitions.

Definition 1.6. Let (X, B) be a log Calabi–Yau pair.

(1) We say that (X, B) is *log rational* if there exists a crepant birational map $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$.
(2) We say that (X, B) is of *cluster type* if there exists a crepant birational map $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ for which $\operatorname{codim}_{\mathbb{P}^n}(\operatorname{Ex}(\phi) \cap \mathbb{G}_m^n) \geq 2$. [4]

We say that a normal projective variety X is *log rational* (resp. *of cluster type*) if there exists a boundary B such that (X, B) is log rational (resp. of cluster type).

Rationality problems are among the most classic problems in algebraic geometry. Then, it is natural to understand whether a rational variety is indeed log rational. We propose the following conjecture (see, e.g., [4, 5]).

Conjecture 1.7. *Let (X, B) be a log Calabi–Yau pair of dimension n. Assume that the following two conditions hold:*

(1) *we have that $\mathcal{D}(X, B) \simeq_{\rm PL} \mathbb{S}^{n-1}$, and*
(2) *every dlt strata of (X, B), including X itself, are rational varieties.*[5]

Then, the pair (X, B) is log rational.

In what follows, we call the previous the *log rationality conjecture*. Both conditions Conjecture 1.7.(1)-(2) are necessary conditions for a pair to be log rational. There are examples in the literature showing that both conditions are necessary (see, e.g., [9] and [14,]). Conjecture 1.7 is known in dimension 2 due to the work of Gross, Hacking, and Keel [7] and for pairs of the form $(\mathbb{P}^3, B)$ due to the work of Ducat [4]. The following seems to be the most reasonable question in this direction.

Question 1.8. *Does the log rationality conjecture hold for pairs (T, B_T) with T a toric Fano 3-fold?*

On the other hand, understanding cluster type pairs (even conjecturally) is still an open problem. It seems that the condition of being cluster type is much more subtle and depends on the topology and singularities of $X \setminus B$.

[4]If instead we impose $\operatorname{codim}_X(\operatorname{Ex}(\phi^{-1}) \cap (X \setminus B)) \geq 2$, then we recover a characterization of toric varieties.

[5]This means that in a suitable dlt modification (partial resolution) are rational varieties.

2. The birational complexity

The birational complexity is an invariant introduced by Mauri and the author to study dual complexes. The birational complexity of (X,B) is the minimum among the complexities of crepant birational models of (X,B). In a few words, the birational complexity measures how far is (X,B) from being birational to a toric pair. Formally speaking, we define

$$c_{\rm bir}(X,B) := \inf\{c(X',B') \mid \ (X',B') \simeq_{\rm cbir} (X,B) \text{ and } B' \geq 0\}.$$

In [14, Lemmatta 2.30-2.32], Mauri and the author proved that the birational complexity is always a minimum, even if B has real coefficients. The following is a connection between log rationality and the birational complexity (see, e.g.)

Theorem 2.1. *Let (X,B) be a log Calabi–Yau pair of index one. Then, the pair (X,B) is log rational if and only if it has birational complexity zero.*

The birational complexity of a n-dimensional log Calabi–Yau pair (X,B), with X Fano, is always contained in the interval $(0,2n)$. If the pair (X,B) has index one, then the birational complexity is an integer in this interval. However, it is unclear which values are achieved in each dimension.

Question 2.2. *Determine which positive integers appear as $c_{\rm bir}(X,B)$ where X is a Fano variety and B is a 1-complement.*

With some extra hypotheses, we expect that the value 1 does not appear. More precisely, we predict that the following question has a positive answer.

Question 2.3. *Let (X,B) be a log Calabi–Yau pair of index one, coregularity zero, and complexity one. Then, the pair is log rational.*

If we restrict ourselves to n-dimensional log Calabi–Yau pairs (X,B) of index one and coregularity zero, then the birational complexity is contained in $\{0,1,\dots,n\}$. Such pairs with birational complexity n are called pairs of *maximal birational complexity*. There are two natural ways to construct log Calabi–Yau pairs of maximal birational complexity. The first is to take finite quotients of log Calabi–Yau pairs of higher birational complexity; the second is to construct log Calabi–Yau pairs structures on birationally superrigid varieties. The first method works in arbitrary dimension (see, e.g., [14, Example 9.1]) while the second method is known to work in dimension three (see [9]). It would be interesting to generalize the second method to arbitrary dimensions.

Question 2.4. *Can we construct examples in each dimension n of birationally superrigid Fano varieties X and log Calabi–Yau pairs (X,B) of coregularity zero and birational complexity n?*

In particular, the following problem is still open.

Problem 2.5. *In each dimension $n \geq 3$, construct a log Calabi–Yau pair (X,B) satisfying the following conditions:*

1. *the index of (X,B) is one,*
2. *the pair (X,B) has coregularity zero, and*
3. *every finite cover of (X,B) has birational complexity n.*

We finish this section with a short problem which shows that the birational complexity is not easy to compute even in the simplest cases.

Problem 2.6. *Let X_d by a n-dimensional Fano hypersurface of degree d. Let B_d be the intersection to X of the sum of $(n+1-d)$ general hyperplanes. Compute the value $c_{\rm bir}(X_d,B_d)$.*

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