Warm-up

1. Show that \( f(x) = x^3 - 3x^2 + 4x - 1 \) has exactly one zero in \( \mathbb{R} \), and it lies in \((0, 1)\). (Soln: \( f(0) = -1, 0 < 1 = f(1) \) and \( f \) is cont., so by IVT \( \exists x \in (0, 1) \) s.t. \( f(x) = 0 \). Also, \( f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \), so if \( f(x) = 0 \) for some \( x \neq 5 \), then by MVT \( \exists \eta \in (x, 5) \) v(5, \( \eta \))
   s.t. \( f'(\eta) = 0 = \frac{f(5) - f(\theta)}{\theta - 5} \), contradiction.)

2. Show that \( f(x) = e^x - \cos(x) - 3 \) has exactly one zero in \( \mathbb{R} \), and it lies in \((\frac{\pi}{3}, \frac{\pi}{2})\). (Soln: for \( x \leq 0 \), \( f(x) = e^x - \cos(x) - 3 \leq 1 - \cos(x) - 3 = -2 - \cos(x) < 0 \),
   so \( f \) has no zeros in \((\cos, 0)\).
   \( f(0) = -3 \neq 0 \), \( f'(x) = e^x + \sin(x) > 1 + 1/2 > 0 \) for \( x > 0 \), which implies (by Rolle's) that any zero
   of \( f \) must be unique, \( f(\frac{\pi}{3}) = 2.81 - 3.5 < 2.81 \cdot 3.5 < (2.81)(1.2) - 3.5 = 2.36 - 3.5 = -1.4 < 0 \),
   \( f(\frac{\pi}{6}) = 2.71^\frac{1}{5} - 3 = (2.71)^{\frac{1}{5}} - 3 > (2.7)(1.6) - 3 = 4.32 - 3 = 1.32 > 0 \), so by IVT,
   \( f \) has a zero in \((\frac{\pi}{6}, \frac{\pi}{3})\).

3. Suppose \( f, g : [0, 1] \rightarrow \mathbb{R} \) are continuous, \( f(0) = -5 = g(1) \), and \( f(1) = 5 = g(0) \). Prove that \( \exists x \in [0, 1] \) with \( f(x) = g(x) \).
   (Soln: set \( h : [0, 1] \rightarrow \mathbb{R} \), \( h(x) = g(x) - f(x) \). Then \( h \)
   is cont. on \([0, 1] \), \( h(0) = g(0) - f(0) = 10 \), and \( h(1) = g(1) - f(1) = -10 \).
   So by IVT \( \exists x \in (0, 1) \) s.t. \( 0 = h(x) = g(x) - f(x) \). Rearrange.)

Root-finding

Given some function \( f \), how can we find (or approximate) roots of \( f ? \) → Bisection, Regula-Falsi, Newton's method, Secant method algorithms.
**Bisection Method**

**Given:** \( f: [a, b] \rightarrow \mathbb{R} \) cont., \( f(a) \cdot f(b) < 0 \).

Set \( a_0 = a, b_0 = b, c_0 := \frac{a_0 + b_0}{2} \).

If \( f(c_0) = 0 \), done; root is \( c_0 \).

Else if \( f(a_0) \cdot f(c_0) < 0 \), set \( a_1 = a_0, b_1 = c_0 \).

else, set \( a_1 = c_0, b_1 = b_0 \).

\( \vdots \) repeat! This generates sequence \( \{c_n\}_{n=0}^\infty \) with

\[ |c_{m-n} - c_n| = \frac{1}{2} |c_{m-n} - c_{m-1}|, \text{ so for } m > n, \text{ we get} \]

\[ |c_m - c_n| \leq \sum_{k=n}^{m-1} |c_{k+1} - c_k| \leq \sum_{k=n}^{m-1} \frac{1}{2^k} |c_n - c_{m-n}| \]

\[ = \frac{(b-a)}{2^{m-n}} = \frac{b-a}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ so } \{c_n\} \text{ is Cauchy.} \]

By completeness of \( \mathbb{R} \), \( \exists c^* \in [a, b] \) with \( c_n \rightarrow c^* \).

\( f(c^*) > 0 \) implies \( \exists S > 0 \) s.t. \( \forall \epsilon > 0 \), \( \exists \delta > 0 \) s.t. \( |x - c^*| < \delta \rightarrow |f(x)| < \frac{|f(c^*)|}{2} \), so \( f(x) > \frac{f(c^*)}{2} > 0 \). But this contradicts that we can find \( a_n, b_n \in (c^*-\delta, c^*+\delta) \) (for large enough \( n \in \mathbb{N} \)) with \( f(a_n) \cdot f(b_n) < 0 \).

We get a similar contradiction assuming \( f(c^*) < 0 \).

Thus \( f(c^*) = 0 \), so \( c^* \) is a zero of \( f \) and the algorithm is justified.

Easier: \( \forall n \in \mathbb{N} \cup \{0\} \), **IVT guarantees \( \exists x^*_n \in [a_n, b_n] \) with \( f(x^*_n) = 0 \).

So, \( |c_n - x^*_n| \leq \frac{b_n - a_n}{2^n} = \frac{b_o - a_o}{2^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty. \)

Thus, \( c_n \), the \( n \)-th iterate produced by the bisection algorithm, is within \( \frac{b-a}{2^n} \) of some root in \([a, b] \).
Regula Falsi ("false position")

\[ f: [a, b] \to \mathbb{R} \text{ cont., } f(a) \cdot f(b) < 0. \]

Set \( a_0 = a, \ b_0 = b \).

For \( j = 0, 1, 2, \ldots, N; \)

Set \( c_j = \) (root of secant line through \( (a_j, f(a_j)), (b_j, f(b_j)) \)) = \( a_j - \frac{(b_j - a_j) f(a_j)}{f(b_j) - f(a_j)} \).

If \( f(c_j) = 0 \), done; root is \( c_j \).
Else if \( f(a_j) \cdot f(c_j) < 0 \), set \( a_{j+1} = a_j, b_{j+1} = c_j \).
Else set \( a_{j+1} = c_j, b_{j+1} = b_j \).

End.

This algorithm produces \( n \)th iterate \( c_n \), and we can use IVT to assert that \( c_n \) is within \( [b_n - a_n] \) of a root of \( f \). But...

no good way to estimate how fast \( [b_n - a_n] \) shrinks to 0, so this method isn't as "robust" as bisection. It works pretty well when \( f \) is nearly linear though.

Newton's method \( f: \mathbb{R} \to \mathbb{R}, \) differentiable. Starting iterate \( p_0 \in \mathbb{R}.

Idea: follow the tangents.

tangent line to \( f \) at \( (p_n, f(p_n)) \):

\[ y - f(p_n) = f'(p_n)(x - p_n). \]

\( \Rightarrow \) zero: \( x = p_n - \frac{f(p_n)}{f'(p_n)} \).

So, take \( p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \).

This is Newton's algorithm, and it generates a seq. \( \{p_n\} \) of (Newton) iterates.
Secant Method
Similar to Newton's, but use secant lines instead of tangent lines.

\( f: \mathbb{R} \to \mathbb{R} \), starting iterates \( p_0, p_1 \in \mathbb{R}, p_0 \neq p_1 \).

Secant line through \((p_0, f(p_0))\) and \((p_1, f(p_1))\):

\[
y - f(p_0) = \left( \frac{f(p_1) - f(p_0)}{p_1 - p_0} \right) (x - p_0), \quad \text{root: } x = p_0 - \left( \frac{p_1 - p_0}{f(p_1) - f(p_0)} \right) f(p_0)
\]

So, given last two iterates \( p_{n-1}, p_n \), set

\[
p_{n+1} = \frac{p_{n-1} f(p_n) - p_n f(p_{n-1})}{f(p_n) - f(p_{n-1})} \quad \text{This produces iterates } (p_n)_{n=0}^{\infty}.
\]

Note: different from Regula-Falsi, since only last two iterates are used. Regula-Falsi can "hold on" to old iterates. Secant method should be viewed as a modification of Newton's method rather than Regula-Falsi.

Next time, we'll analyze the convergence rate of bisection algorithm & Newton's method, discuss convergence of sequences in more depth, and get into fixed-point iteration.

Problems

1. Draw a curve \( y = f(x) \) over an interval \([0, 1]\) for which:
   a) Regula-Falsi beats bisection
   b) bisection beats Regula-Falsi
   c) bisection beats Newton's method
   d) Newton beats bisection
   e) Newton beats secant
   f) Secant beats Newton

2. Find \( n \) large enough so that bisection applied to \( f(x) = \sin(x) \) over the interval \([3, 4]\) finds an approximate root to within \( 10^{-3} \) of the actual root \( \pi \), \( (n = 9) \).

3-5 see next page!
Let \( m \in \mathbb{N} \) be some natural number. By applying Newton's method to the function \( f(x) = \frac{1}{x} - m \), devise an iterative procedure to find \( \frac{1}{m} \) that does not involve any divisions. Then find all possible initial iterates \( p_0 \in \mathbb{R} \) for which the algorithm converges; i.e., \( p_n \to \frac{1}{m} \) as \( n \to \infty \).

**Solution:** Newton's algorithm gives us the iterative procedure

\[
p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n - m}{-\frac{1}{m^2}} = p_n + (p_n - mp_n^2) = 2p_n - mp_n^2.
\]

The error obeys a recursive relationship:

\[
|p_{n+1} - \frac{1}{m}| = \left|2p_n - mp_n^2 - \frac{1}{m} - m\left(\frac{1}{m^2}\right)^2\right| = \left|2\left(p_n - \frac{1}{m}\right) - m\left(p_n^2 - \frac{1}{m^2}\right)\right|
\]

\[
= \left|p_n - \frac{1}{m}\right| - 2 - m\left(p_n + \frac{1}{m}\right)
\]

\[
= \left|p_n - \frac{1}{m}\right| - 2 - m\left(p_n + \frac{1}{m}\right)
\]

Clearly, if \( p_0 \neq \frac{1}{m} \), then \( p_n \) will never be exactly \( \frac{1}{m} \). Suppose \( |p_0 - \frac{1}{m}| < \frac{1}{m} \). Then if \( |p_{k+1} - \frac{1}{m}| < \min(\frac{1}{m}, |p_k - \frac{1}{m}|) \), we get

\[
|p_{k+1} - \frac{1}{m}| = m\left|p_k - \frac{1}{m}\right|^2 = m\left|p_k - \frac{1}{m}\right|\left|p_k - \frac{1}{m}\right| < \frac{1}{m}
\]

and

\[
|p_{k+1} - \frac{1}{m}| = m\left|p_k - \frac{1}{m}\right|^2 < m\left(\frac{1}{m^2}\right) = \frac{1}{m}, \text{ so}
\]

\[
|p_{k+1} - \frac{1}{m}| < \min(\frac{1}{m}, |p_k - \frac{1}{m}|).
\]

The base case \( |p_1 - \frac{1}{m}| = m\left|p_0 - \frac{1}{m}\right|^2 < \min(\frac{1}{m}, |p_0 - \frac{1}{m}|) \)

is satisfied, so if \( \alpha < |p_0 - \frac{1}{m}| < \frac{1}{m} \), then

\[
|p_{n+1} - \frac{1}{m}| \text{ strictly decreases with } n. \text{ Since } |p_n - \frac{1}{m}| > 0 \text{ for all } n, \text{ and bounded,}
\]

\[
\text{monotonic sequences converge in } \mathbb{R}, \text{ we get that } \lim_{n \to \infty} |p_n - \frac{1}{m}| = L \text{ exists. Then,}
\]

\[
L = \lim_{n \to \infty} |p_{n+1} - \frac{1}{m}| = \lim_{n \to \infty} m^2\left|p_n - \frac{1}{m}\right|^2 = m\left(\lim_{n \to \infty} |p_n - \frac{1}{m}|\right)^2 = mL^2, \Rightarrow L = 0 \text{ or } \frac{1}{m}.
\]

Since \( |p_0 - \frac{1}{m}| < \frac{1}{m} \) and \( |p_{n+1} - \frac{1}{m}| \) is decreasing, the only possible limit is \( \lim_{n \to \infty} |p_n - \frac{1}{m}| = 0 \).

Thus \( p_n \to \frac{1}{m} \) as \( n \to \infty \). If \( |p_0 - \frac{1}{m}| > \frac{1}{m} \), then \( |p_n - \frac{1}{m}| \) implies \( |p_{n+1} - \frac{1}{m}| = m|p_n - \frac{1}{m}|^2 \)

\[
\geq m\left(\frac{1}{m}\right)^2 = \frac{1}{m}, \text{ so induction shows that } |p_n - \frac{1}{m}| > \frac{1}{m} \text{ for all } n \to \infty.
\]
4. Let $m \in \mathbb{N}$, $a > 0$. By applying Newton's method to the function $f(x) = x^m - a$, devise an algorithm to approximate $a^{1/m}$. Show that the root errors obey the recursive relationship $|p_{m+1} - p| = \frac{|p_m - p|^2}{2|p_m|}$ in the case $m=2$, where $p = a^{1/m}$ is (an) actual root of $f$.

**Solt:** Newton's method gives

$$p_{m+1} = p_m - \frac{f(p_m)}{f'(p_m)} = p_m - \frac{p_m^m - a}{m p_m^{m-1}} = p_m(1 - \frac{1}{m}) + \frac{a}{m p_m^{m-1}}.$$ 

If $m=2$, this gives "Heron's algorithm",

$$p_{n+1} = \frac{p_n^2 + a}{2p_n}.$$ 

If $m=2$, then $p = a^{1/2} = \sqrt{a}$. (Note: the recursive error relationship still holds, even if we set $p = -a^{1/2}$.)

$$p_{n+1} - p = \left(\frac{p_n^2 + a}{2p_n}\right) - \left(\frac{p_n^2 + a}{2p_n}\right) = \frac{p_n - p}{2} + \frac{a(p - p_n)}{2p_n p_n} = \frac{(p_n - p)(1 - \frac{a}{p_n p_n})}{2} = \frac{(p_n - p)^2}{2p_n},$$

then take magnitudes to get the recursive error relationship.

5. Prove that if $p_{m+1} = \frac{p_n^2 + a}{2p_n}$, where $a > 0$, then $p_n \to \sqrt{a}$ if $p_0 > 0$ and $p_n \to -\sqrt{a}$ if $p_0 < 0$.

**Solt:** Case 1: $p_0 > 0$. Then $(p_{k+1} \geq \sqrt{a}) \iff (\frac{p_n^2 + a}{2p_n} \geq \sqrt{a}) \iff (p_k - \sqrt{a} \geq 0)$.

If $p_{m+1} \leq p_k$, then $p_{m+1} = \frac{p_n^2 + a}{2p_n} \leq p_k \iff \frac{p_n^2 + a}{2p_n} \leq p_k$.

So $p_k > 0$ implies $p_{m+1} \geq \sqrt{a}$, so $\forall m \in \mathbb{N}$, $p_n \geq \sqrt{a}$. Similarly, $(p_{m+1} \leq p_k) \iff (\frac{p_n^2 + a}{2p_n} \leq p_k) \iff (a \leq p_n^2) \iff (p_k \geq \sqrt{a})$, so $(p_n)_{n=1}^{\infty}$ is decreasing and bounded below. So $\lim_{n \to \infty} p_n = L$ exists and $L = \lim_{n \to \infty} p_{m+1} = \lim_{n \to \infty} \frac{p_n^2 + a}{2p_n} = \frac{L^2 + a}{2L}$ (and $L \geq \sqrt{a}$).

Then $L^2 = L^2 + a$, so $L^2 = a$, so $L = \pm \sqrt{a}$. As $L \geq \sqrt{a}$, we know $L = \sqrt{a} = \sqrt{\frac{1}{m}} p_0$.

Case 2: $p_0 < 0$. Set $q_0 = -p_0 > 0$. Put $q_{m+1} = \frac{q_n^2 + a}{2q_n}$, so $\lim_{n \to \infty} q_n = -\sqrt{a}$ by case 1. $(p_k = -q_k) \implies (q_{m+1} = \sqrt{a})$ so $p_n = -q_n$ for all $n \in \mathbb{N}$. Thus $\lim_{n \to \infty} p_n = -\sqrt{a}$. 

6. Set $a_0 = 0$, $a_{n+1} = \sqrt{2 + a_n}$. Show that $(a_n)_{n=0}^\infty$ converges in $\mathbb{R}$, and find what it converges to.

**Solution:** $a_{k+1}$ implies $a_{k+1} \geq \sqrt{2} \geq 0$, so all $a_n$'s are nonnegative, so the sequence is well-defined.

Let's find all possible limits of $(a_n)_{n=0}^\infty$.

Suppose $\lim_{n \to \infty} a_n = L \in \mathbb{R}$ exists. Then

$$L = \lim_{n \to \infty} (a_{n+1}) = \lim_{n \to \infty} (\sqrt{2 + a_n}) = \sqrt{2 + \lim_{n \to \infty} a_n} = \sqrt{2 + L}.$$ 

Apply indices shift, definition, limit laws, as $g(x) = \sqrt{2 + x}$ is continuous.

Then $L^2 = 2 + L$, so $L^2 - L - 2 = 0$, so $L = 2$ or $-1$.

$a_n \geq 0 \forall n$ implies $L \geq 0$, so that leaves $L = 2$.

Thus if $\lim_{n \to \infty} (a_n)$ converges, then it must converge to $L = 2$.

$(a_{k+1} \leq 2) \iff (\sqrt{2 + a_k} \leq 2) \iff (2 + a_k \leq 4) \iff (a_k \leq 2)$.

Since $a_0 = 0 \leq 2$, induction gives us that $a_n \leq 2$ for all $n \in \mathbb{N}$.

$(a_{k+1} \geq a_k) \iff (\sqrt{2 + a_k} \geq a_k) \iff (2 + a_k \geq a_k^2) \iff (a_k^2 - a_k - 2 \leq 0) \iff (a_k - 2)(a_k + 1) \leq 0$.

So $(a_n)_{n=0}^\infty$ is increasing and bounded above; so $\lim_{n \to \infty} (a_n)$ exists, and by the work above, we know $\lim_{n \to \infty} (a_n) = 2$.