Note: other frames of reference are possible. e.g. the angle $\theta$ could be measured from the positive $\hat{i}$ axis rather than the negative $\hat{j}$ axis.

$$\mathbf{x} = L \hat{\mathbf{r}} = x \hat{i} + y \hat{j}.$$ 

Since $||\hat{\mathbf{r}}||=1$ and $\hat{i} & \hat{j}$ are orthonormal, we have $L = \sqrt{x^2 + y^2}$.

$$\begin{cases} \hat{r} = (\sin \theta) \hat{i} - (\cos \theta) \hat{j} \\ \hat{\theta} = (\cos \theta) \hat{i} + (\sin \theta) \hat{j} \end{cases} \iff \begin{cases} \hat{i} = (\sin \theta) \hat{r} + (\cos \theta) \hat{\theta} \\ \hat{j} = (-\cos \theta) \hat{r} + (\sin \theta) \hat{\theta} \end{cases}$$

Since $\mathbf{x} = L \hat{\mathbf{r}} = (L \sin \theta) \hat{i} - (L \cos \theta) \hat{j}$, we must have $\begin{cases} L \sin \theta = x \\ L \cos \theta = -y \end{cases}$.

If we were given $x$ & $y$, we could easily figure out $L$ & $\theta$, and thus also $\hat{r} & \hat{\theta}$. As $x$ & $y$ are unknown here, and the problem asks us to use polar coordinates rather than rectangular, $x$ & $y$ will not be used.

Note that $\hat{r} & \hat{\theta}$ change in time according to $\theta = \dot{\theta}(t)$, unlike $\hat{i} & \hat{j}$. But rather than treating $\hat{r} & \hat{\theta}$ as unknowns, we should treat $\theta$ & $L$ as unknowns, and $\hat{r} & \hat{\theta}$ will follow.

Note: $\frac{d}{dt} \hat{r} = \hat{\theta} \frac{d}{dt} \theta$, $\frac{d}{dt} \hat{\theta} = -\hat{r} \frac{d}{dt} \theta$. So

$$\frac{d}{dt} \mathbf{x} = \dot{x} = (\hat{L} \hat{r} + L \hat{\theta}) \hat{r} + L \hat{\theta} \hat{\theta} = L \hat{\theta} \hat{\theta}$$ is the velocity vector.
14.2 (cont.) b) \( \ddot{\alpha} = \frac{d^2\alpha}{dt^2} = \dddot{x} = (L\dddot{\theta} + L\dot{\theta}\dot{\theta}) \)

\[ = L\dddot{\theta} + L\dddot{\theta} + L\dot{\theta}\dot{\theta} + L\dot{\theta}\dot{\theta} \]

\[ = L\dddot{\theta} + L\dot{\theta}\dot{\theta} + L\dot{\theta}\dot{\theta} + L\dot{\theta}\dot{\theta} - L(\dot{\theta})^2 \]

\[ = (L\dot{\theta} + 2L\dot{\theta})\dot{\theta} + (\dddot{\theta} - L(\dot{\theta})^2) \]

\[ c) \text{ If } L = \text{const. and } \frac{d\theta}{dt} = \dot{\theta} = \text{const., then} \]

\[ L = 0 = \ddot{\theta}, \text{ so } \ddot{\mathbf{v}} = \dddot{x} = L\dot{\theta}\dot{\theta}, \quad \dddot{\alpha} = \dddot{x} = L\dddot{\theta} - L(\dot{\theta})^2 \dddot{\mathbf{r}} = -L\dot{\theta}^2 \]

By integrating \( \dot{\theta} = k \), we get \( \theta = kt + c \). Since \( L \) is constant in time, the motion of the object is circular with constant speed.

It makes sense that the velocity \( \dddot{\mathbf{v}} \) points tangentially to the circle, with no radial component, since the distance of the object to the origin (L) never changes. It makes sense that the acceleration \( \dddot{\alpha} \) points in the \(-\mathbf{r}\) direction, since the object is moving as if it were attached by a string of length \( L \) to the origin, and in that case the only force acting on the object to sustain its circular motion would be from the string tension, which points inward (in the \(-\mathbf{r}\) direction).

14.12 length of \( \mathbf{\hat{r}} \cdot \mathbf{\hat{r}} = \int \mathbf{\hat{r}} \cdot \mathbf{\hat{r}} \). Since this is constant, squaring yields \( \mathbf{\hat{r}} \cdot \mathbf{\hat{r}} = \text{const.} \), and thus differentiating with respect to time yields

\[ (\mathbf{\hat{r}} \cdot \mathbf{\hat{r}})' = \mathbf{\hat{r}}' \cdot \mathbf{\hat{r}} + \mathbf{\hat{r}} \cdot \mathbf{\hat{r}}' = 2 \mathbf{\hat{r}} \cdot \mathbf{\hat{r}}' = 0, \text{ so } \mathbf{\hat{r}} \text{ and } \mathbf{\hat{r}}' \text{ are always orthogonal to each other.} \]
16.2 linearized pendulum: \( \frac{d^2 \theta}{dt^2} = -\frac{g \theta}{L} \), \( \theta(0) = \theta_0 \).

Substitutions: \( \Theta(t) = a \phi(t), \quad t = b \tau \). \( \frac{d \Theta}{dt} (0) = \Omega_0 \).

Then \( \frac{d \phi}{d \tau} = \frac{d \phi}{dt} \cdot \frac{dt}{d \tau} = \frac{1}{a} \frac{d \Theta}{dt} \cdot b = \frac{b}{a} \frac{d \Theta}{dt} \), so

\[
\frac{d^2 \phi}{d \tau^2} = \frac{b^2}{a^2} \frac{d^2 \Theta}{dt^2} \cdot \frac{dt}{d \tau} = \frac{b^2}{a} \frac{d^2 \Theta}{dt^2} = \left( -\frac{gb^2}{aL} \right) \phi
\]

and new initial conditions \( \phi(0) = \frac{\Theta(0)}{a} = \frac{\theta_0}{a}, \quad \frac{d \phi}{d \tau} (0) = \frac{b}{a} \frac{d \Theta}{dt} (0) = \frac{b \Omega_0}{a} \).

Let's set \( b = \sqrt[2]{\frac{1}{3}} \) to simplify the ODE.

At this point we can choose \( a \) to simplify either \( \phi(0) \) or \( \frac{d \phi}{d \tau} (0) \), but not both (since then \( a \) would be overdetermined).

Let's set \( a = \theta_0 \) (another option would be \( a = b \Omega_0 \)). Then our IVP (initial value problem) becomes:

\[
\frac{d^2 \phi}{d \tau^2} = -\phi, \quad \phi(0) = 1, \quad \frac{d \phi}{d \tau} = \frac{\Omega_0}{\theta_0} \sqrt{\frac{L}{g}}.
\]

The only parameter now is \( k = \frac{\Omega_0}{\theta_0} \sqrt{\frac{L}{g}} \).

Nonlinear pendulum: doesn't work, because we would end up getting

\[
\frac{d^2 \phi}{dt^2} = \frac{b^2}{a^2} \frac{d^2 \Theta}{dt^2} = \frac{-gb^2}{aL} \sin \Theta = \frac{-gb^2}{aL} \sin(a \phi)
\]

\[\neq \frac{-gb^2}{L} \sin \phi.\]

In this case, the introduction of the constant \( a \) just complicates the ODE, so it is not helpful. The problem is that \( \sin(a \phi) \) cannot be simplified.
\[ 16.3 \quad [g] = \frac{[\text{force}]}{[\text{mass}]} = \frac{MD/T^2}{M} = \frac{D}{T^2} \]

\[ [\omega_0] = [\frac{d\theta}{dt}] = \frac{[\theta]}{[\text{rad}]} = \frac{1}{T}, \text{since radians are dimensionless} \]

\[ [\frac{\Delta \theta}{\omega_0}] = \frac{[\Delta \theta]}{[\omega_0]} = \frac{\Delta \theta}{\omega_0} = \frac{1}{\frac{1}{T}} = 1, \text{so } \frac{\Delta \theta}{\omega_0} \text{ is dimensionless.} \]

\begin{equation}
\text{Problem 3} \quad m \ddot{v} = mg - kv \frac{\dot{v}}{v} \]
\end{equation}

(a) Equilibria: \( 0 = mg - kv \frac{\dot{v}}{v} \iff \frac{\dot{v}}{v} = \frac{mg}{k} > 0 \),

so \( v \) must be positive, and thus \( v = \sqrt{\frac{mg}{k}} \) is the only equilibrium. \( v > \sqrt{\frac{mg}{k}} \) implies \( \frac{\dot{v}}{v} > \frac{mg}{k} \) implies \( \dot{v} < 0 \), and \( v < \sqrt{\frac{mg}{k}} \) implies \( \frac{\dot{v}}{v} < \frac{mg}{k} \) implies \( \dot{v} > 0 \). Thus not only is \( v = \sqrt{\frac{mg}{k}} \) stable; it is a global sink, which means regardless \( v \) of the initial condition, \( v \to \sqrt{\frac{mg}{k}} \) as \( t \to \infty \).
Problem 4

A bead on a horizontal wire

The tension force on m from the spring is \( k \left( \sqrt{x^2 + h^2} - l \right) \). By looking at triangle side ratios, we can see that the horizontal component of this force is \( \left( \frac{-x}{\sqrt{x^2 + h^2}} \right) \cdot k \left( \sqrt{x^2 + h^2} - l \right) \). Careful with signs: if the spring is stretched, the force must oppose \( x \), and if the spring is compressed, the force must be in the same direction as \( x \). Thus, neglecting force terms in the vertical direction (which must necessarily cancel out), we get

\[
m\ddot{x} = k x \left( \frac{\ell}{\sqrt{x^2 + h^2}} - 1 \right) - b \dot{x}.
\]

b) Equilibria: if \( x \equiv \text{const.} \), then \( \dot{x} = \ddot{x} = 0 \).

\[
0 = k x \left( \frac{\ell}{\sqrt{x^2 + h^2}} - 1 \right) - 0 \iff \text{either } x = 0 \text{ or } \frac{\ell}{\sqrt{x^2 + h^2}} = 1.
\]

\[
\frac{\ell}{\sqrt{x^2 + h^2}} = 1 \iff x = \pm \sqrt{x^2 - h^2}.
\]

Thus if \( l > h \), then we have three equilibria: \( x = 0, \pm \sqrt{x^2 - h^2} \), and if \( l \leq h \), then \( x = 0 \) is the only equilibrium. Stability: in general, we could analyze the stability of each equilibrium by writing the system as \( \ddot{v} = \dot{x} \), \( \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = F(x, v) \), and analyzing the eigenvalues of \( \text{Jac}(F) \) for each equilibrium point \((x_E, v_E)\). More on this later!
Problem 4 (cont.)

b) (cont.) Intuition tells us that if $l > h$, then $x_E = \pm \sqrt{l^2 - h^2}$ should both be stable and $x_E = 0$ should be unstable; and if $l \leq h$, then $x_E = 0$ should be stable. Note that since $\dot{x}$ depends on both $x$ & $\ddot{x}$, a simple 2D phase plane of $\ddot{x}$ vs. $x$ won’t help us determine possible solution trajectories, and thus this method cannot be used to determine the stability of equilibrium points. Graphing $\ddot{x}$ vs. $x$ & $\dot{x}$ in 3D would enable us to draw possible solution trajectories, but it would be very difficult (and not rigorous) to do this by hand.

c) If $m \neq 0$, then any miniscule deviation of $\dot{x}$ away from 0 will cause drastic acceleration, since $m \ddot{x} = \ddot{x}$ and $m$ is infinitesimally small. If $\ddot{x}$ is positive, then $\dot{x} < \frac{kx}{b} \left( \frac{1}{\sqrt{x^2 + h^2}} - 1 \right)$, and the drastic acceleration will force $\dot{x}$ to increase to $\frac{kx}{b} \left( \frac{l}{\sqrt{x^2 + h^2}} - 1 \right)$; if $\ddot{x}$ is negative, then $\dot{x} > \frac{kx}{b} \left( \frac{l}{\sqrt{x^2 + h^2}} - 1 \right)$, and the drastic (negative) acceleration will force $\dot{x}$ to decrease to $\frac{kx}{b} \left( \frac{l}{\sqrt{x^2 + h^2}} - 1 \right)$. Thus we may say that as $m \neq 0$, the ODE describing how $x$ changes in time is given by $\dot{x} = \frac{kx}{b} \left( \frac{l}{\sqrt{x^2 + h^2}} - 1 \right)$, which is what we happen to get when simply setting $m = 0$ in the 2nd order ODE from part (a). Note that this would not be valid if $b$ were negative, for then we would get runaway velocity to $\pm \infty$. 
Problem 4 (cont., 2)

c) (cont.) If \( l > h \), then \( x_E = 0, \pm \sqrt{l^2 - h^2} \), and if \( l \leq h \), then \( x_E = 0 \), all just as in part \( (b) \). But now, since \( x \) depends only on \( x \), we can easily determine the stability of each equilibrium point \( x_E \).

Case 1: \( l > h \). Since we only need to know the sign of \( \dot{x} \) for each \( x \) value, there's no need to graph a phase plane. We can instead use a schematic like so:

\[
\begin{align*}
\dot{x} & : + 0 - 0 + 0 - \\
x & : \left\{ \begin{array}{c}
\frac{l}{\sqrt{x^2 + h^2}} \\
0 \\
\frac{Jk}{\sqrt{x^2 + h^2}}
\end{array} \right.
\end{align*}
\]

and thus we may immediately conclude that \( x_E = 0 \) is unstable (as \( x \) goes from \(-\) to \(+\) there), and \( x_E = \pm \sqrt{l^2 - h^2} \) are both stable (as \( x \) goes from \(+\) to \(-\) at those equilibria).

Graphically, \( x \) is unstable. We could also set

\[
f(x) = \frac{kx}{b}\left(\frac{l}{\sqrt{x^2 + h^2}} - 1\right),\text{ and check that } f'(0) > 0, f'\left(\pm \sqrt{l^2 - h^2}\right) < 0.
\]

Case 2: \( l \leq h \). If \( x \neq 0 \), then \( l \leq h = Jh^2 < Jx^2 + h^2 \), so \( \frac{l}{\sqrt{x^2 + h^2}} < 1 \). Thus:

\[
\begin{align*}
\dot{x} & : + 0 - \\
x & : \left\{ \begin{array}{c}
\frac{l}{\sqrt{x^2 + h^2}} \\
0
\end{array} \right.
\end{align*}
\]

so \( x_E = 0 \) is stable.
Problem 5

\[ \begin{cases} \varepsilon \dddot{x} + \ddot{x} + x = 0 \\ x(0) = 1 \\ \dot{x}(0) = 0 \end{cases}, \quad \varepsilon > 0. \]

a) Characteristic equation: \( \varepsilon r^2 + r + 1 = 0 \), so \( r = \frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon} \).

If \( \varepsilon > \frac{1}{4} \), then

\[ x(t) = e^{\text{Re}(r)t} \left( c_1 \cos(\text{Im}(r)t) + c_2 \sin(\text{Im}(r)t) \right) \]

Put \( \omega = \frac{\sqrt{4\varepsilon - 1}}{2\varepsilon} \), \( \alpha = \frac{1}{2\varepsilon} \). Then

\[ x(t) = e^{\alpha t} \left( c_1 \cos(\omega t) + c_2 \sin(\omega t) \right) \]

so

\[ \begin{cases} x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ 2c_1 + \omega c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -\frac{\omega}{\omega} = \frac{1}{\sqrt{4\varepsilon - 1}} \end{cases} \]

If \( \varepsilon = \frac{1}{4} \), then \( r = -2 \) is a double root, so

\[ x(t) = c_1 e^{-2t} + c_2 te^{-2t} = e^{-2t} (c_1 + c_2 t) \]

\[ \dot{x}(t) = e^{-2t} (c_2 - 2c_1 - 2c_2 t) \]

so

\[ \begin{cases} x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 - 2c_1 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 2 \end{cases} \]

If \( \varepsilon < \frac{1}{4} \), then set \( r_1 = \frac{-1 - \sqrt{1 - 4\varepsilon}}{2\varepsilon} \), \( r_2 = \frac{-1 + \sqrt{1 - 4\varepsilon}}{2\varepsilon} \).

\[ x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

\[ \dot{x}(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} \]

so

\[ \begin{cases} x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 r_1 + c_2 r_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{r_2}{r_2 - r_1} = \frac{1}{2} - \frac{1}{2\sqrt{\varepsilon - \varepsilon}} \\ c_2 = \frac{-r_1}{r_2 - r_1} = \frac{1}{2} + \frac{1}{2\sqrt{\varepsilon - \varepsilon}} \end{cases} \]
Problem 5 (cont.)

b) $\varepsilon << 1$: last case from part (a), so

\[ \lim_{\varepsilon \to 0} r_1 = \lim_{\varepsilon \to 0^+} \frac{-1 - \sqrt{1 - 4\varepsilon}}{2\varepsilon} = 0, \]

and \( \lim_{\varepsilon \to 0} r_2 = \lim_{\varepsilon \to 0^+} \frac{-1 + \sqrt{1 - 4\varepsilon}}{2\varepsilon} = -\frac{4}{2\sqrt{1 - 4\varepsilon}} = -1 \).

So \( \frac{1}{r_1} \) approaches 0 and \( \frac{1}{r_2} \) approaches -1.

c) $\varepsilon_0 = 0.001$, so $1 - 4\varepsilon_0 = 0.996$. Since $(1 - 2\varepsilon_0)^2 = 1 - 4\varepsilon_0 + 4\varepsilon_0^2 > 1 - 4\varepsilon_0$

and $(1 - 3\varepsilon_0)^2 = 1 - 6\varepsilon_0 + 9\varepsilon_0^2 < 1 - 6\varepsilon_0 + \varepsilon_0 = 1 - 5\varepsilon_0 < 1 - 4\varepsilon_0$,

we know that $0.997 < \sqrt{1 - 4\varepsilon_0} < 0.998$, so $e(-1.998, -1.997)$

\[ r_1 = \frac{-1 - \sqrt{1 - 4\varepsilon_0}}{2\varepsilon_0} e^{(-1.999, -1.998.5)} \]

and \( r_2 = \frac{-1 + \sqrt{1 - 4\varepsilon_0}}{2\varepsilon_0} e^{(-1.5, -1)} \).

Also, \( 0 > c_1 = \frac{1}{2} - \frac{1}{2\sqrt{1 - 4\varepsilon_0}} > \frac{1}{2} - \frac{1}{2(0.997)} = \frac{1}{2} (1 - \frac{1}{1 - 3\varepsilon_0}) \)

\[ = \frac{1}{2} (1 - [1 + 3\varepsilon_0 + (3\varepsilon_0)^2 + (3\varepsilon_0)^3 + \ldots]) > \frac{1}{2} (1 - [1 + 3\varepsilon_0 + \varepsilon_0]) \]

\[ = -2\varepsilon_0 = -0.002 \]

and \( c_2 = 1 - c_1 e^{(1, 1.002)} \).

\[ X(t) = c_1 e^{rt} + c_2 e^{rt} \]

Note: \( -\frac{1}{r_1} \approx 0.001 \) and \( -\frac{1}{r_2} \approx 1 \).
Problem 5 (cont.2)

d) \[
\begin{align*}
\begin{cases}
x' + x &= 0 \\
x(0) &= 1
\end{cases} &\Rightarrow \\
\begin{cases}
r + 1 &= 0 \\
r &= -1
\end{cases} &\Rightarrow \\
x(t) &= c_1 e^{-t} \\
1 &= c_1
\end{align*}
\]

The solution to this IVP is \( x(t) = e^{-t} \).

If \( t \approx 0 \) (\( t \approx 0 \)), then both solutions are near \( x = 1 \), even though the solution to the \( E_0 = 0.01 \) problem has slope \( \approx 0 \) near \( t = 0 \) and the solution here has slope \( \approx -1 \) near \( t = 0 \). The approximation is a good 0th order approximation, but would be at least a good 1st order approximation if the initial conditions for the 2nd order IVP were \( \begin{cases} x(0) = 1 \\ x'(0) = 0 \end{cases} \) instead.

So considering this, the two solutions differ enough near \( t = 0 \) to be considered poor approximations of each other.

Now, considering \( t \gg 1 \)...

\[
\lim_{t \to \infty} \frac{e^{\frac{-t}{c_1 e^{rt} + c_2 e^{rt}}}}{t^{\infty}} = \lim_{t \to \infty} \frac{e^{\frac{-t}{c_1 + c_2 e^{rt}}}}{t} = \infty, \text{ so the solution}
\]

to the 1st order ODE is (relatively) much larger than the solution to the 2nd order ODE, even though both approach 0. So again, for \( t \gg 1 \), the approximation isn't great.

If, however, we were to fix \( t \in (0, \infty) \) and send \( \varepsilon \to 0 \), we would get \( \lim_{E_0 \to 0} \frac{e^{\frac{-t}{c_1 e^{rt} + c_2 e^{rt}}}}{t} = 1 \), by a similar argument to the above, so \( x_e(t) \to x(t) \) as \( \varepsilon \to 0 \) at least.
Problem 6 \[ \begin{cases} \dot{N} = rN \left(1 - \frac{N}{K}\right) \\ N(0) = N_0 \end{cases} \]

a) \( N \) usually represents \# of individuals in a population, so we may consider this to be a dimensionless parameter,
\[ [N] = 1, [r] = \frac{\dot{N}}{N} = \frac{1}{\tau}, \text{ so } [r] = \frac{1}{\tau}. \]

Regardless of what units we choose to express \( N \) in, \([K] = [N] = [N_0]\) and \([r] = \frac{1}{\tau}\).

b) \( \begin{cases} N = ax \\ t = b\tau \end{cases} \) for \( a \) & \( b \) to be determined, nonzero.

Then \( \frac{1}{\tau} N = a \frac{d}{dt} x \), and \( \frac{dt}{d\tau} = b \), so

\[ \frac{dx}{dt} = \frac{1}{a} \frac{dN}{d\tau}, \frac{dN}{d\tau} = \frac{b}{a} \frac{dt}{d\tau}, \text{ chain rule} \]

\[ \frac{dx}{dt} = \frac{b}{a} rN \left(1 - \frac{N}{K}\right) = \frac{b}{a} r(ax) \left(1 - \frac{(ax)}{K}\right) \]

\[ = b r x \left(1 - \frac{a x}{K}\right), \text{ so setting } b = \frac{1}{r}, a = K \]

gives \( \begin{cases} \frac{dx}{d\tau} = x \left(1 - \frac{x}{K}\right) \\ x(0) = \frac{N(0)}{K} = \frac{N_0}{K} = x_0 \end{cases} \). \( \text{Note: if } r = 0 \text{ or } N_0 = 0, \text{ then } N = N_0, \) so no need to simplify the ODE.

By similar reasoning, \( a = N_0, b = \frac{1}{r} \), so \( \begin{cases} \frac{d}{d\tau} u(1 - \frac{N}{K}) u \quad N = N_0 u \\ u(0) = 1, \quad t = \tau / r \end{cases} \)

(c) will be solved faster by numerical algorithms on computer, because fewer multiplications, but when converting back to original variables, you end up doing those multiplications anyways.
Extra Problems

E2  a) \( x = 3x^2 - 27 = 3(x - 3)(x + 3) \), so
\[ x_E = \pm 3. \]
\[ x': + 0-0+ \]
so \( x_E = 3 \) stable, \( x_E = -3 \) unstable.

b) \( \dot{x} = x - x^3 = x(1-x)(1+x) \), so \( x_E = 0, \pm 1 \)
\[ x': + 0-0+0- \]
\[ x': \leftarrow 0 \rightarrow \]
\( x_E = \pm 1 \) stable, \( x_E = 0 \) unstable.

E3 \( x = f(x) = xh(x) \), \( h' \) exists near \( x = 0 \), \( h(0) \neq 0 \).
So \( x_E = 0 \) is an equilibrium point and
\[ f'(x_E) = f'(0) = [h(x) + xh'(x)]_{x=0} = h(0) \neq 0. \]
\( x_E = 0 \) stable \( \iff \) \( h(0) < 0 \).

E4 \[ \left[ \frac{L}{g t^2} \right] = \frac{D}{(\frac{P}{t^2}) t^2} = 1, \quad [\Theta] = 1, \quad \text{and} \quad [\frac{T}{mg}] = \frac{MD/\ell^2}{MD/\ell^2} = 1. \]
Furthermore, \( L = \left( \frac{L}{g t^2} \right) g t^2 \), \( g = g \), \( m = m \), \( T = (\frac{T}{mg}) mg \), 
\( \Theta = (\Theta) \), and \( t = t \), so each parameter can be expressed as one of the dimensionless groups times a product of powers of the primary dimensions.
Miscellaneous Problems

M2. \( \dot{x} = -\frac{3}{l} x \)

a) Extremely light mass attached to weightless, swinging rod (pendulum), with air friction causing linear damping \( (m \ddot{x} = -\frac{3}{l} x - bx) \). Since \( \ddot{x} \) and \( x \) have opposite signs here, an argument like that given for problem 4C shows that \( x \approx -\frac{3}{bl} x \).

b) \( \int \frac{dx}{\sin x} = -\int dt \), so \( \int \frac{-\csc^2(x) - \csc(x)\cot(x)}{\csc(x) + \cot(x)} dx = \int dt \),

or: \( \ln |\csc(x) + \cot(x)| = t + C \), \( x(0) = x_0 \).

c) \( x_0 = \frac{\pi}{4} ; \ln (J^2 + 1) = 0 + C \). Since \( x(0) = \frac{\pi}{4} > 0 \) and equilibrium points \( x = 0, \pi \) can't be crossed, \( x(t) \in (0, \pi) \).

So \( \csc(x) + \cot(x) = (1 + J^2) e^t \).

\[
\begin{align*}
\csc(x) + \cot(x) &= \frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \\
&= \frac{1 + \cos(x/2)}{\sin(x/2)} = \frac{(\cos^2(x/2) + \sin^2(x/2)) + (\cos^2(x/2) - \sin^2(x/2))}{2\sin(x/2)\cos(x/2)} \\
&= \frac{\cos(x/2)}{\sin(x/2)} = \frac{1}{\tan(x/2)} , \text{ so } \tan(x/2) = \frac{e^t}{(1+J^2)},
\end{align*}
\]

and thus \( x = 2\arctan\left( \frac{e^t}{1+J^2} \right) \).

d) \( x \to x_0 \) as \( t \to \infty \), as expected, since \( \dot{x} < 0 \) for \( x \in (0, \pi) \) and \( x = 0 \) is the greatest equilibrium less than \( x_0 = \frac{\pi}{4} \).

e) \( x = 2\arctan\left( \frac{e^t}{A} \right) + 2n\pi \), where \( A = \csc(\theta) + \cot(\theta) \) and \( n = \frac{x_0 - 2\arctan(1/\theta)}{2\pi} \). If \( \sin(x_0) = 0 \), then \( x \equiv x_0 \).
M2 (cont.)

f) \( x_E = m \pi, \ m \in \mathbb{Z} \) are the equilibria.

\[
f(x) = -\sin x; \quad f'(x_E) = -\cos (m \pi) = \begin{cases} -1, & m \text{ even} \smallskip \\ 1, & m \text{ odd} \end{cases}
\]

\( x_E = m \pi \) (stable)

\( m \) (even)

M3

\( v(t) \geq 0 \) implies \( v = |v| \), so ODE becomes

\[
m |v| = mg - kv^2.
\]

Then perform separation of variables yields

\[
\int \frac{mdv}{mg - kv^2} = \int dt.
\]

\[
\frac{m}{mg - kv^2} = \frac{1}{g - \frac{k}{m}v^2} = \frac{1}{(g - \frac{k}{m}v)(g + \frac{k}{m}v)}
\]

\[
= \frac{1}{2g} \left( \frac{1}{g - \frac{k}{m}v} + \frac{1}{g + \frac{k}{m}v} \right)
\]

\[
\Rightarrow \frac{1}{2} \int \frac{m}{Kg} \left[ \ln |g + \frac{k}{m}v| - \ln |g - \frac{k}{m}v| \right] = t + C
\]

\[
= \ln \left| \frac{g + \frac{k}{m}v}{g - \frac{k}{m}v} \right|
\]

\[
\Rightarrow \frac{g + \frac{k}{m}v}{g - \frac{k}{m}v} = Ae^{2tC}, \quad \Rightarrow |v| = \frac{ma}{k} \left( \frac{Ae^{2tC} - 1}{Ae^{2tC} + 1} \right)
\]

where \( A = \frac{g + \frac{k}{m}v}{g - \frac{k}{m}v} \). Thus

\[
\lim_{t \to \infty} v(t) = \frac{ma}{k}, \quad \text{same as what we got in problem 3.}
\]