Homework #1 Solutions

2.1 a) net force acting on \( m_i: \mathbf{F}_i + \mathbf{F}_i \) for \( i = 1, 2 \). Thus \( m_i \ddot{x}_i = \mathbf{F}_i + \mathbf{F}_i \) for \( i = 1, 2 \). Taking derivatives of \( \dot{x}_c = \frac{m_1 \dot{x}_1 + m_2 \dot{x}_2}{m_1 + m_2} \), we get

\[
\ddot{x}_c = \frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2} = \frac{\mathbf{F}_1 + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_2}{m_1 + m_2} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{m_1 + m_2},
\]

since \( \mathbf{F}_1 = -\mathbf{F}_2 \)

and thus \( m_i \ddot{x}_c = \mathbf{F} \), where \( m = m_1 + m_2 \) and \( \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \).

b) \( \dot{x}_c = \frac{m_1 \dot{x}_1 + m_2 \dot{x}_2}{m_1 + m_2} = \left( \frac{m_1}{m_1 + m_2} \right) \dot{x}_1 + \left( \frac{m_2}{m_1 + m_2} \right) \dot{x}_2 \). Set

\[
\dot{x} = \frac{m_1}{m_1 + m_2}.
\]

Then \( \dot{x} \in (0, 1) \) since \( m_1 > 0 < m_2 \), and

\[
\ddot{x}_c = \dot{x} \ddot{x}_1 + (1 - \dot{x}) \ddot{x}_2 = \ddot{x}_2 + \dot{x} (\ddot{x}_1 - \ddot{x}_2).
\]

Letting \( L(\lambda) = \lambda \ddot{x}_2 + (1 - \lambda) \ddot{x}_1, \) where \( \lambda \in \mathbb{R} \), this maps the real line to the line in \( \mathbb{R}^n \) passing through \( \ddot{x}_1 \) & \( \ddot{x}_2 \) (note that \( L \) is affine, \( L(0) = \ddot{x}_2 \), and \( L(1) = \ddot{x}_1 \)). Since \( \dot{x} \in (0, 1) \), we know that \( \ddot{x}_c = L(\dot{x}) \) lies on the line segment connecting \( \ddot{x}_1 \) & \( \ddot{x}_2 \).

4.1 \( x(t) = \text{height @ time } t \).
\[
\dot{x} = -mg \Rightarrow \dot{x} = -gt + C. \quad \dot{x}(0) = V_0 \text{ implies } C = V_0. \text{ Thus }
\]

\[
x = -\frac{1}{2}gt^2 + V_0 t = t(V_0 - \frac{g}{2}t) + D. \text{ We may take } x(0) = 0 = D, \text{ so this is a downward-opening parabola with roots } t = 0, \frac{2V_0}{g} \text{. Thus } x_{\text{max}} = x\left(\frac{V_0}{g}\right) = \frac{V_0^2}{2g}. \]
4.2 \[ x(t) = \text{horizontal position} \]
\[ y(t) = \text{vertical position} \]
\[ \begin{align*}
  x(t) &= 0 \quad \text{(no horizontal forces acting on mass)} \\
  m \ddot{x} &= 0 \quad \text{since } m = C. \quad \text{Since } x(0) = v_0, \ C = m v_0. \\
  \text{Thus } \dot{x} &= v_0 \quad \text{and } x = v_0 t + D. \quad \text{Taking } \\
  x(0) &= 0 \Rightarrow D = 0 \Rightarrow x(t) = v_0 t. \\
  m \ddot{y} &= -mg \quad \text{(gravitational force downward, treating } g > 0) \\
  \Rightarrow \quad \ddot{y} &= -g, \quad \text{so } \dot{y} = -gt + E. \quad \text{Taking } \\
  y(0) &= 0 \quad \text{(table assumed to be level), so } E = 0. \quad \text{Then } y = -\frac{g}{2} t^2 + F. \\
  y(0) &= h, \quad \text{so } F = h. \quad \text{Thus } y(t) = h - \frac{g}{2} t^2.
\end{align*} \]

The mass lands when \( y(t^*) = 0 \), so \( t^* = \sqrt{\frac{2h}{g}} \), and this means \( x(t^*) = v_0 \sqrt{\frac{2h}{g}} \).

Since \( t = \frac{x}{v_0} \), we may write \( y = h - \frac{g}{2} \left( \frac{x}{v_0} \right)^2 = h - \frac{g}{2v_0^2} x^2 \), and so the trajectory the mass takes is a downward-opening parabola.

5.1 Using the trigonometric angle-addition formula \( \sin(a+b) = \sin(a) \cos(b) + \sin(b) \cos(a) \)

gives \( \sin(3t - \pi/2) = \sin(3t) \cos(-\pi/2) + \sin(-\pi/2) \cos(3t) \)
\[ = -\cos(3t) \]

so \( x = 2 \sin(3t - \pi/2) = -2 \cos(3t) \).

5.2 Note: we want to express \( x \) in the form \( A \sin(\omega t + \phi) = A [\sin(\omega t) \cos(\phi) + \sin(\phi) \cos(\omega t)] \), so we need \( \omega = 1 \) in order the period to be correct, and also: \( (A \geq 0) \)
\[
\begin{align*}
  \{ A \cos(\phi) &= 3 \} \quad &\Rightarrow & \quad \{ A^2 (\sin^2 \phi + \cos^2 \phi) = 3^2 + (-1)^2 \} \\
  A \sin(\phi) &= -1 \quad &\Rightarrow & \quad \{ \frac{A \sin \phi}{A \cos \phi} = \frac{-1}{3} \} \\
  \tan \phi &= -\frac{1}{3}.
\end{align*}
\]

Since \( \sin \phi < 0 < \cos \phi \), we know that \( \phi \) must lie in the 4th quadrant. Since \( \arctan(-1/3) \) already lies in the 4th quadrant, there's no need to add \( \pi \) to this. Thus \( A = \sqrt{10} \quad \phi = \arctan(-\frac{1}{3}) \).
5.6 a) \( e^{iwt} = \sum_{n=0}^{\infty} \frac{(iwt)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iwt)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iwt)^{2n+1}}{(2n+1)!} \)

\( = \sum_{n=0}^{\infty} \frac{(i)^n (wt)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(i)^n (wt)^{2n+1}}{(2n+1)!} \)

\( = \sum_{n=0}^{\infty} \frac{(wt)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(wt)^{2n+1}}{(2n+1)!} \)

\( = \cos(wt) + i \sin(wt) \), where the splitting up of the series for \( e^{iwt} \) is justified by the fact that \( e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \) is absolutely convergent (e.g., by ratio test) for all \( z \in \mathbb{C} \).

b) \( e^{-iwt} = e^{(-i)wt} = \cos((-i)wt) + i \sin((-i)wt) \)

\( = \cos(wt) - i \sin(wt) \)

since \( \cos(z) \) is even & \( \sin(z) \) is odd.

5.8 a) \( mx = -kx \Rightarrow \) characteristic equation is \( mr^2 + k = 0 \), so \( r = \pm i \sqrt{\frac{k}{m}} \).

Thus the general solution to the given ODE is \( x(t) = c_1 \cos(wt) + c_2 \sin(wt) \), where \( w = \sqrt{\frac{k}{m}} \).

b) Using \( \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \) yields:

\[ B \cos(wt + \theta_0) = B \left[ \cos(wt) \cos(\theta_0) - \sin(wt) \sin(\theta_0) \right] \]

Matching coefficients, we want

\[ \begin{cases} B \cos\theta_0 = c_1 \\ -B \sin\theta_0 = c_2 \end{cases} \]

\( B \cos\theta_0 = c_1 \iff B = \frac{c_1}{\cos\theta_0} \) \( \iff \tan\theta_0 = \frac{c_2}{c_1} \).

Since this is always possible (even if \( c_1 = 0 \) \( [B = \lvert c_2 \rvert, \theta_0 = -\frac{c_2}{c_1} \frac{\pi}{2}] \) or \( c_1 = 0 \), \( c_2 \neq 0 \) \( [B = 0 = \theta_0] \)), and clearly any \( (B, \theta_0) \) gives rise to a (unique) \( (c_1, c_2) \), we are done.
6.2  a) \[ [k] = \frac{[\text{force}]}{[\text{distance}]} = \frac{MD}{D} = \frac{M}{T^2} \quad \text{and} \quad [m] = M, \]
where \( M = \text{mass} \), \( D = \text{distance} \), and \( T = \text{time} \).

\[ \Rightarrow \text{if} \quad y = \sqrt{\frac{m}{k}}, \quad \text{then} \quad [y] = \sqrt{\frac{M}{M/T^2}} = T. \]

b) \[ y = c\sqrt{\frac{m}{k}}, \quad \text{where} \quad c \in \mathbb{R} \quad \text{is dimensionless}. \]

7.1 \[ 12 = f = \frac{1}{T} = \frac{1}{\left(\frac{2\pi}{\omega}\right)} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \int k \quad \text{so} \quad k = (2\pi \cdot 12)^2 m = 576\pi^2 m, \]

where \( [m] = \text{kilograms} \), \( [k] = \text{kg} \cdot \text{sec}^{-2} \).

If a new weight \( m' \) is placed on the same spring, then the new resulting frequency is
\[ f' = \frac{1}{2\pi} \int k' \quad \text{so} \quad f' = 12 \sqrt{\frac{m}{m'}}. \]

So if, in particular, \( m' = 4m \), then \( f' = 12/2 = 6 \text{ sec}^{-1} \).

7.2  By equating the spring force to gravitational force (required for equilibrium), we get \( kd = mg \). If all of our constants are in terms of mks units, then \( d = 0.25 \),
so \( k = \frac{mg}{d} = 40mg \), and thus the natural frequency of oscillation is
\[ f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \int k \quad \text{so} \quad f = \frac{1}{2\pi} \cdot \sqrt{\frac{m}{m}}. \]

(Same frequency, whether spring is oriented horizontally or vertically).
9.1 \[ \begin{aligned} x(0) &= 0, \quad \dot{x}_1(0) = \gamma \\
\dot{x}_2(0) &= \beta, \quad \dot{x}_2(0) = \delta \end{aligned} \]

By eqn. 9.6, \( \ddot{z} = -k \left( \frac{1}{m_1} + \frac{1}{m_2} \right) z \), so
\[ z = c_1 \cos(\omega t) + c_2 \sin(\omega t), \]
where \( \omega = k \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \).

By eqn. 9.5, \( z = x_2 - x_1 - l \), so
\[ x_2 - x_1 = \lambda + c_1 \cos(\omega t) + c_2 \sin(\omega t), \]
and
\[ \dot{x}_2 - \dot{x}_1 = -c_1 \omega \sin(\omega t) + c_2 \omega \cos(\omega t), \]
so using the given initial conditions, this means \( c_1 = \beta - \omega \lambda, \quad c_2 = \frac{\delta - \gamma}{\omega} \).

By eqn. 9.4, \( m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0 \), so \( m_1 \dot{x}_1 + m_2 \dot{x}_2 = \text{const} = m_1 \gamma + m_2 \delta \), and thus
\[ m_1 \dot{x}_1 + m_2 \dot{x}_2 = (m_1 \gamma + m_2 \delta) t + \text{const}, \]
so
\[ m_1 \dot{x}_1 + m_2 \dot{x}_2 = (m_1 \gamma + m_2 \delta) t + m_1 \lambda + m_2 \beta \]

Now, combining
\[ x_1 = \left( \frac{1}{m_1 + m_2} \right) \left( (m_1 \gamma + m_2 \delta) t + m_1 \lambda + m_2 \beta - m_1 \lambda - m_2 \gamma \cos(\omega t) - m_2 \delta \sin(\omega t) \right) \]
\[ x_2 = \left( \frac{1}{m_1 + m_2} \right) \left( (m_1 \gamma + m_2 \delta) t + m_1 \lambda + m_2 \beta + m_1 \lambda + m_2 \gamma \cos(\omega t) + m_2 \delta \sin(\omega t) \right) \]
where \( \omega, c_1, \) and \( c_2 \) are given above.

9.2
\[ \begin{aligned} x(0) &= 0 \\
\dot{x}_1(0) &= m_0 g + k(x_2 - x_1 - l) \\
\dot{x}_2(0) &= m_0 g - k(x_2 - x_1 - l) \end{aligned} \]

\[ \begin{aligned} m_1 \ddot{x}_1 + m_2 \ddot{x}_2 &= (m_1 + m_2) g, \\
\dot{z} &= x_2 - x_1 - l, \\
\ddot{z} &= -k \left( \frac{1}{m_1} + \frac{1}{m_2} \right) z \end{aligned} \]

so the oscillation of the masses about their equilibrium w.r.t. the spring remains exactly the same as before, but now the entire system is accelerating downwards at a rate of \( g \).
9.3 a) The springs can be stretched when the system is in equilibrium \((d > l_1 + l_2)\), or they can be compressed... \((d < l_1 + l_2)\). \((d = l_1 + l_2) \iff \text{(system @ equilibrium when the springs are relaxed)}\). 

b) \[ 0 = -k_1(x-l_1) + k_2(d-x-l_2) \]

or:

\[ x_E = \frac{k_1l_1 + k_2(d-l_2)}{k_1+k_2} \] 

When \(k_1 = k_2\), \(l_1 = l_2\): \(x_E = \frac{d}{2}\), as expected.

c) \[ m\ddot{x} = -k_1(x-l_1) + k_2(d-x-l_2) \]

\[ = -(k_1+k_2)(x-x_E) \] 

so the mass exhibits SHM about its equilibrium.

d) \[ T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k_1+k_2}{m}}} = 2\pi \sqrt{\frac{m}{k_1+k_2}} \] 

e) \( T \) doesn't depend on \(d\).

9.4 a) \[ 0 = -k_1(x-l_1) - k_2(x-l_2), \text{so } x_E = \frac{k_1l_1 + k_2l_2}{k_1+k_2} \]

b) \[ m\ddot{x} = -k_1(x-l_1) - k_2(x-l_2) \]

\[ = -(k_1+k_2)(x-x_E) \] 

so \(x\) exhibits SHM about its equilibrium, \(x_E\).

c) \[ T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k_1+k_2}{m}}} = 2\pi \sqrt{\frac{m}{k_1+k_2}} \] 

d) new \(l = \frac{k_1l_1 + k_2l_2}{k_1+k_2}\); new \(k = k_1 + k_2\).
9.6

\[ \begin{align*}
& a) \left\{ \begin{array}{l}
o = -k(x-l) + k(d-x-y-l) \\
o = k(d-x-y-l) - k(y-l)
\end{array} \right. \\
\leftrightarrow \left\{ \begin{array}{l}
o = -2kx - ky + kl \\
o = -kx - 2ky + kl
\end{array} \right.
\]

\[ \Rightarrow \begin{array}{c}
x = \frac{y}{k} = \frac{1}{2}, \text{ makes sense.}
\end{array} \]

\[ b) \begin{align*}
\{ m \ddot{x} &= -kx + k(d-x-y) \\
\dot{y} &= k(d-x-y) - ky
\} \quad \Rightarrow m(d-x-y) &= k(x+y) - 2k(d-x-y)
\end{align*} \]

\[ = k(3x+3y-2d) = 3k \left( [d-x-y] - \frac{d}{3} \right), \]

\[ \text{So } (d-x-y) = \text{dist. b/w masses exhibits SHM about its equilibrium at } \frac{d}{3}. \text{ Alternatively, we could set } z = d-x-y, z_0 = \frac{d}{3}. \]

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{3k}}. \]

\[ c) \quad m(x-y) = -k(x-y), \text{ so } (x-y) \text{ exhibits SHM (about its equilibrium at } 0), \text{ with } T = 2\pi \sqrt{\frac{m}{k}} \neq 2\pi \sqrt{\frac{m}{3k}}. \]

\[ d) \quad \text{If } d-x-y = \text{const.} = \frac{d}{3}, \text{ (its equilibrium, the only possible constant value, by part (b)), then}
\]

\[ \left\{ \begin{array}{l}
m \ddot{x} &= -kx + k \left( \frac{d}{3} \right) = -k(x-\frac{d}{3}) \\
\dot{y} &= k \left( \frac{d}{3} \right) - ky = -k(y-\frac{d}{3})
\end{array} \right. \text{, and } \frac{2d}{3} = x+y, \text{ so}
\]

\[ y = \frac{d}{3} - (x-\frac{d}{3}) \text{ at all times, and thus } x \text{ exhibits SHM about its equilibrium at } \frac{d}{3}, \text{ and } y \text{ exhibits symmetrical SHM about its equilibrium at } \frac{d}{3}. \]

\[ T = 2\pi \sqrt{\frac{m}{k}}. \text{ (visually, the masses move with identical SHM)} \]

\[ e) \quad \text{If } x = y \text{ for all time, then}
\]

\[ \left\{ \begin{array}{l}
m \ddot{x} &= -3k(x-\frac{d}{3}) \\
\dot{y} &= -3k(y-\frac{d}{3})
\end{array} \right. \]

\[ \text{so } x \text{ & } y \text{ exhibit identical SHM about their equilibria at } \frac{d}{3}, \]

\[ \text{with period } T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{m}{3k}}} = 2\pi \sqrt{\frac{m}{3k}}. \text{ (visually, the masses move with symmetrical SHM)} \]
10.1  a) \( \alpha < 0 \) in order for the force of friction to impede the object's motion rather than accelerate it.

b) \( [\alpha] = \frac{[\text{force}]}{[\text{velocity}]^2} = \frac{MD/T^2}{D^3/T^3} = \frac{MT}{D^2} \)

10.3 \[ \frac{[c]}{[\text{velocity}]} = \frac{MD/T^2}{D/T} = \frac{M}{T} \]

10.8 \[ m \ddot{x} = -mg - c \dot{x} \]

\[ \frac{c}{m} \]

\[ \dot{y} = -\frac{c}{m} y, \text{ so } y = Ae^{\frac{-ct}{m}} = x + \frac{mg}{c} \]

\[ \dot{x} = x + \frac{mg}{c} + Ae^{\frac{-ct}{m}} \]

where \( A \) will depend on initial velocity. \( \lim_{t \to \infty} x(t) = -\frac{mg}{c} + A \lim_{t \to \infty} e^{\frac{-ct}{m}} = 0 \)

so the object approaches a terminal velocity of \( -\frac{mg}{c} \) as \( t \to \infty \).

Note: \( c > 0 \) is required for the friction force to impede travel.

10.9  a) \( m \ddot{x} = -\beta \dot{x}^n \) (assuming \( \dot{x} > 0 \)), \( 0 < n < 1 \). Use \( y = \dot{x} \).

\[ m \frac{dy}{dt} = -\beta y^n, \text{ so } \int \frac{dy}{y^n} = \frac{-\beta}{m} \int dt, \text{ or } \frac{1}{1-n} y^{1-n} = \frac{-\beta}{m} t + C, \]

so \( \dot{x} = \frac{1}{1-n} \left( C - \frac{\beta(1-n)}{m} t \right) ^{1-n} \), and if \( \dot{x}(0) = v_0 \), then \( C = v_0^{1-n} \), or: \( \dot{x} = \left( v_0^{1-n} - \frac{\beta(1-n)}{m} t \right) ^{1-n} \), so \( x = x = \frac{m}{(2-n)\beta} \left( v_0^{1-n} - \frac{\beta(1-n)}{m} t \right) ^{\frac{2-n}{1-n}} \)

where \( D \) is computed using the initial position (irrelevant).

The object ceases to move when \( \dot{x} = 0 \), which will happen at time \( t^* = \frac{m v_0^{1-n}}{\beta(1-n)} \). The particle will travel a distance \( \Delta x \) \( x(t^*) - x(0) = \frac{m v_0^{1-n}}{(2-n)\beta} \).
10.9 (cont.)

a) \[ \frac{\mu v_0^n}{(1-n)^2} \]

\[ x = \frac{\mu v_0^n}{(1-n)^2} t \]

b) if \( n=1 \), then \( \dot{x} = -\beta x \). char. eqn.: \( m \ddot{x} + \beta r = r (mr + \beta) x \).

So \( \lambda = 0 \) or \( -\beta \frac{m}{\lambda} \). Thus \( x = c_1 + c_2 e^{-\beta t / m} \), where

\[ x(0) = x_0 = c_1 + c_2, \quad x'(0) = v_0 = -\beta c_2, \quad \Rightarrow \]

\[ x = \left( x_0 + \frac{mv_0}{\beta} \right) - \frac{mv_0}{\beta} e^{-\beta t / m} \]

\[ \text{exponential decay,} \]

\[ x = \frac{\mu y_0}{\beta} \]

11.1

\[ [c^2] = [c]^2 = \left( \frac{\text{[force]}}{\text{[velocity]}} \right)^2 = \left( \frac{MD}{D/T} \right)^2 = \frac{M^2}{T^2} \]

\[ [mk] = [m][k] = M \cdot \left( \frac{\text{[force]}}{\text{[distance]}} \right) = \frac{M^2 D}{T^2} = \frac{M^2}{T^2} \]

11.2 Solution has terms of type \( e^{rt} \), so roots \( r \) must have \( [r] = \frac{1}{T} \) in order for \( rt \) to be dimensionless.

Since \[ [c] = \frac{M}{T} = \left[ \sqrt{c^2 - 4mk} \right] = \sqrt{[c^2 - 4mk]} = \sqrt{\frac{M^2}{T^2}} = \frac{M}{T}, \]
we get \[ [r] = \frac{\frac{M}{T}}{M} = \frac{1}{T}, \] as expected.
12.4 (note: \(c<0\) is unrealistic, so unless specifically stated otherwise, always assume \(c>0\))

\[m \ddot{x} = -kx - c \dot{x} \iff m \ddot{x} + cx + kx = 0.\]

char. eqn.: \(mr^2 + cr + k = 0 \iff r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.

\[\text{If } c^2 - 4mk > 0, \text{ then both roots are positive (since } -c > 0,\text{ and } c = |c| = \sqrt{c^2} > \sqrt{c^2 - 4mk}), \text{ so the general solution is } x = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \text{ The roots will be distinct (since } c^2 - 4mk > 0), \text{ so say } 0 < r_1 < r_2. \text{ Then } \lim_{t \to \infty} x(t) = \begin{cases} \infty & \text{if } c_1 > 0, \\ -\infty & \text{if } c_2 < 0. \end{cases}\]

\[\text{If } c^2 - 4mk = 0, \text{ then } r = \frac{-c}{2m} > 0 \text{ is a double root, so } x = c e^{r t} + c_2 t e^{r t}, \text{ and } |x| \to \begin{cases} \infty & \text{if } c_2 > 0, \\ -\infty & \text{if } c_2 < 0. \end{cases}\]

\[\text{If } c^2 - 4mk < 0, \text{ then the roots are complex conjugates of each other: } r = \frac{-c}{2m} \pm i \frac{\sqrt{4mk - c^2}}{2m}, \text{ so } x = e^{-\frac{ct}{2m}} (c \cos(\omega t) + c_2 \sin(\omega t)), \text{ where } \omega = \frac{\sqrt{4mk - c^2}}{2m}.\]

Here, \(|x|\) periodically gets very large, since \(e^{-\frac{ct}{2m}} \to 0\) as \(t \to \infty\) (\(c<0\)).

These three cases can in fact be handled in one fell swoop by saying: at least one root has positive real part. This indicates that the solution will blow-up in time.

\[A = \sqrt{c_1^2 + c_2^2}.\]
13.1 \[ c^2 > 4mk \] (now, back to assuming \( c > 0 \))

a) \( m\ddot{x} + c\dot{x} + kx = 0 \); char. eqn. \( m\dddot{r}^2 + cr + k = 0 \), so

\[ r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]. Since \( c^2 - 4mk > 0 \), we get two distinct, real roots, \( r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \), \( r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \).

Since \(-c < 0\) and \( c = |c| = \sqrt{c^2} > \sqrt{c^2 - 4mk} \), we have \( r_1 < r_2 < 0 \), and \( x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \). Say \( x(0) = x_0 \), \( \dot{x}(0) = v_0 \). Then

\[
\begin{cases}
  c_1 + c_2 = x_0 \\
  c_1 r_1 + c_2 r_2 = v_0
\end{cases} \quad \Rightarrow \quad \begin{cases}
  c_1 = \frac{r_2 x_0 - v_0}{r_2 - r_1} \\
  c_2 = \frac{v_0 - r_1 x_0}{r_2 - r_1}
\end{cases}.
\]

The mass possesses equilibrium if:

\[ 0 = x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \iff \quad -c_1 e^{r_1 t} = c_2 e^{r_2 t} \]

\[ \iff \quad \frac{-c_1}{c_2} = e^{(r_2 - r_1)t} \]

\[ \iff \quad \frac{v_0 - r_2 x_0}{v_0 - r_1 x_0} = e^{(r_2 - r_1)t} \]

Since \( (r_2 - r_1) > 0 \), we know that \( g(t) := e^{(r_2 - r_1)t} \) obeys:

\[ \lim_{t \to -\infty} g(t) = 0, \quad \lim_{t \to \infty} g(t) = \infty, \quad g(0) = 1, \quad g \text{ is continuous, and } g \text{ is strictly monotonically increasing.} \]

Thus \( \bigstar \) has exactly one solution \( t \in [0, \infty) \) if \( \left( \frac{v_0 - r_2 x_0}{v_0 - r_1 x_0} \right) \in [1, \infty) \),

exactly one solution \( t \in (-\infty, 0) \) if \( \left( \frac{v_0 - r_2 x_0}{v_0 - r_1 x_0} \right) \in (0, 1) \),

and no real solutions \( (t \in \mathbb{R}) \) if \( \left( \frac{v_0 - r_2 x_0}{v_0 - r_1 x_0} \right) \in (-\infty, 0] \).

(If \( v_0 - r_1 x_0 = 0 \), no real solutions.)
13.1 (cont.) a) (cont.) In all of these cases, since both $r_1$ & $r_2$ are negative, eventually $x$ decays exponentially to 0, which means the mass decays to equilibrium, since here $x$ represents displacement from equilibrium.

Furthermore, since $x = c_1r_1e^{rt} + c_2r_2e^{rt}$, then by similar reasoning,

$$x = 0 \iff \left( \frac{r_1}{r_2} \cdot \frac{V_0 - r_2x_0}{V_0 - r_1x_0} = e^{(r_2-r_1)t} \right)$$

so again $x$ will cross 0 at most once, which implies $x$ is eventually monotonic, regardless of the initial conditions. In fact, if $x$ crosses 0 at some time $t^*$, then $0 < \frac{V_0 - r_2x_0}{V_0 - r_1x_0} = e^{(r_2-r_1)t^*}$, and so (since $\frac{r_1}{r_2} > 1$) $x = 0$ when

$$0 < \left( \frac{r_1}{r_2} \right) \cdot \frac{V_0 - r_2x_0}{V_0 - r_1x_0} = e^{(r_2-r_1)t} = \left( \frac{r_1}{r_2} \right) e^{(r_2-r_1)t^*} > e^{(r_2-r_1)t^*} > 0$$

which means $x$ crosses 0 at a time $t > t^*$ greater than the time $t^*$ at which $x$ crosses 0, as expected ($\frac{r_1}{r_2} > 1$).

Returning to the discussion of whether or not $x$ crosses 0.

Since $r_1 < r_2 < 0$, we get that:

$$\begin{align*}
E(-\infty, r_1) &\iff \frac{V_0}{x_0} \in (r_1, r_2) \iff \frac{V_0/x_0 - r_2}{V_0/x_0 - r_1} \in (1, \infty) \iff x(t^*) = 0 \text{ for one } t^* \in (0, \infty) \\
V_0 \downarrow E[r_1, r_2] &\iff \frac{V_0}{x_0} \in [r_1, r_2] \iff \frac{V_0/x_0 - r_2}{V_0/x_0 - r_1} \in [0, 0] \iff x \text{ never crosses 0} \\
E(r_2, \infty) &\iff \frac{V_0}{x_0} \in (r_2, \infty) \iff \frac{V_0/x_0 - r_2}{V_0/x_0 - r_1} \in (0, 1) \iff x(t^*) = 0 \text{ for one } t^* \in (-\infty, 0).
\end{align*}$$

(note: if $x_0 = 0$, then $x(t^*)$ for $t^* = 0$).

b) If $x_0 > 0$, then for $x$ to cross 0 @ a positive time $t^*$, we need $V_0 < x_0r_1$, by part (a). The threshold value is $V_0 = x_0r_1$. 
13.2 \hspace{1cm} c^2 = 4mk

a) \hspace{1cm} r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m}, \text{ double root of char. eqn.}

Thus general solution is: \hspace{1cm} x = c_1 e^{rt} + c_2 te^{rt}, \text{ where } r < 0.

Since \hspace{1cm} \lim_{t \to \infty} x(t) = c_1 \lim_{t \to \infty} e^{rt} + c_2 \lim_{t \to \infty} \frac{t}{e^{rt}} = 0 + c_2 \lim_{t \to \infty} \frac{1}{te^{-rt}} = 0,

x \text{ decays (roughly) exponentially to 0 in time, which means the mass exponentially decays to its equilibrium in time.}

(0 = x) \iff (-c_1 e^{rt} = c_2 te^{rt}) \iff (-\frac{c_1}{c_2} = t).

Since \hspace{1cm} \begin{cases} c_1 = xo \\ c_1 r + c_2 = v_0 \end{cases}, \text{ we have } \begin{cases} c_1 = xo \\ c_2 = v_0 - rx_0 \end{cases}.

If \hspace{1cm} c_2 = v_0 - rx_0 = 0, \text{ then } x \text{ never hits 0, or is identically 0}.

If \hspace{1cm} \frac{v_0}{x_0} < r, \text{ then } \frac{1}{r - \frac{v_0}{x_0}} > 0, \text{ so } \frac{x_0}{rx_0 - v_0} > 0, \text{ so } x(t) = 0 \text{ for exactly one } t^*(> 0).

If \hspace{1cm} \frac{v_0}{x_0} > r, \text{ then } \frac{1}{r - \frac{v_0}{x_0}} < 0, \text{ so } \frac{x_0}{rx_0 - v_0} < 0, \text{ so } x(t) = 0 \text{ for exactly one } t^*(< 0).

Since \hspace{1cm} x = (c_1 r + c_2) e^{rt} + (c_2 t) e^{rt} \text{ has the same form as } x, \text{ we conclude that } x \text{ is eventually one sign only, so}

x \text{ is eventually monotonic.}

\hspace{1cm} x \uparrow \downarrow \t  \downarrow \uparrow \t

b) \hspace{1cm} by \text{ part(a)}, \text{ we get a positive root } t^* > 0 \text{ for } x(t^*) = 0 \iff \text{ \text{iff}} v_0 < rx_0, \text{ so } v_0 = rx_0 \text{ is the threshold value, just as before.}

Miscellaneous problems \rightarrow \text{ read section 39 of the textbook}