Math 142 Homework 5

1. 54) 2, 3, 5, 11, 12

2. For an n-th degree polynomial \( a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n = 0 \), the Hurwitz matrices are

\[
H_1 = \begin{bmatrix} a_1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix}, \ldots
\]

where the \( a_i \)'s in the above are 0 when \( i > n \). The Routh-Hurwitz criterion is that all roots \( \lambda \) have negative real parts iff the determinant of each \( H_i \) is positive, \( 1 \leq i \leq n \).

(a) Derive the Routh-Hurwitz conditions for \( \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \).

(b) Find the matrix \( H_4 \) and derive the Routh-Hurwitz conditions for a 4-th degree polynomial (\( a_0 = 1 \)).

3. 3 populations compete for resources. \( x \) and \( z \) compete with \( y \), but not with each other.

\[
\begin{align*}
\frac{dx}{dt} &= x(4 - 2x - y) \\
\frac{dy}{dt} &= y(5 - x - y - z) \\
\frac{dz}{dt} &= z(6 - 2y - z)
\end{align*}
\]

In the parts below, \( x_c, y_c, z_c \) are positive, but change from part to part.

(a) Find the critical point \( (x_c, y_c, z_c) \). Use Routh-Hurwitz to show it is unstable.

(b) Show that the equilibrium \( (0, y_c, z_c) \) is unstable.

(c) Show that the equilibrium \( (x_c, 0, z_c) \) is stable.

(d) Show that there is no equilibrium \( (x_c, y_c, 0) \).

(e) Show that the equilibrium \( (0, y_c, 0) \) is stable.

(f) All other equilibria are unstable. What does this say about the populations as \( t \to \infty \)?

4. \( z \) is a predator of \( x, y \) and \( z \) compete for resources. resulting system is

\[
\begin{align*}
\frac{dx}{dt} &= x(4 - x - z) \\
\frac{dy}{dt} &= y(3 - 2y - z) \\
\frac{dz}{dt} &= z(-1 + x - y - z)
\end{align*}
\]

(a) Find the critical point \( (x_c, y_c, z_c) \) and use the Routh-Hurwitz criterion to show it is stable.

(b) What animals do you think \( x, y, \) and \( z \) are?
5. Let \( u(x, t) \geq 0 \) be a population density for \(-\infty < x < \infty\).

(a) Suppose that \( u \) moves with flux \( Q(x, t) = -D \frac{\partial u}{\partial x} \). Explain what this means physically about how the population is moving.

(b) Show from the equation in a) and Conservation of Mass that \( \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \). (Remember Conservation of Mass from the beginning of the course, where \( u \) was called \( \rho \).)

(c) Show that \( u(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \) is a solution to the equation in b). (Write it as \( t^{-\frac{1}{2}} e^{-\frac{1}{4t}x^2t^{-1}} \) and use product rule.)

(d) Set \( D = .25 \) and graph the solution in c) for \( t = 1, \ t = 4, \) and \( t = 9. \) Is this consistent with your answer from part a)?

6. Combining the above with our recent study of population modeling,

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + ru \left( 1 - \frac{1}{K} u \right)
\]

models a population that moves in space, but also reproduces with logistic growth. This is called the Fisher equation, which is an example of a ‘Reaction-Diffusion’ equation.

(a) Let \( u(x, t) = U(x - ct) \), and \( V = U' \). Derive a 2-dimensional system of ODE’s for \( U \) and \( V \).

(b) Find the critical points, and the Jacobian at each critical point.

(c) Show that if \( r > 0 \) the critical point at \((0, K)\) is always a saddle.

(d) Suppose that the critical point at \((0, 0)\) is a spiral in (complex eigenvalues). Sketch the general solution to the system for \((U, V)\) in the \(UV\)-plane.

(e) Sketch \( U(x - ct) \) for the solution in d) for \( t = 0, 1, 2. \) Explain why this is NOT a physically realistic solution.

(f) Suppose that the critical point at \((0, 0)\) is a sink (real eigenvalues). Sketch the general solution to the system for \((U, V)\) in the \(UV\)-plane.

(g) Sketch \( U(x - ct) \) for the solution in f) for \( t = 0, 1, 2. \) Explain why this IS a physically realistic solution.

(h) Compute the values of \( c \) for which the eigenvalues of the critical point \((0, K)\) are real. (Hence the solutions in f,g). This gives the ‘minimum wave speed’ for the traveling wave \( U(x - ct) \).

(i) How does everything in the above change if \( r < 0? \)

7. (a) Let \( N \) be the Numic population and \( P \) the Prenumic population in the great basin. Assuming 1-dimensional model for the space variable \( x \), use the competing species equations given in the article to write the reaction-diffusion equations for \( \frac{dN}{dt} \) and \( \frac{dP}{dt} \). Use the specific numbers for the parameters that are given in the article.
(b) Find and solve the competing species equation

\[
\frac{dN}{dt} = N(r_1 - \alpha N - \beta P)
\]

\[
\frac{dP}{dt} = P(r_2 - \gamma N - \delta P)
\]

corresponding the the parameters from above. Sketch the general solution.

(c) Suppose \( N = U(x - ct) \), \( V = U' \), \( P = W(x - ct) \), \( Z = W' \). Find the 4 \( \times \) 4 system of ode’s for \( U, V, W, Z \) corresponding to the reaction-diffusion equation for \( N, P \).

(d) Find all critical points for the equations in c).

(e) Find a critical point of the form \((a, 0, 0, 0)\). Find conditions for which the eigenvalues are real. If the eigenvalues are real, how many are negative?

(f) Find a critical point of the form \((0, 0, b, 0)\). Find conditions for which the eigenvalues are real. If the eigenvalues are real, how many are negative?

(g) Assume the two critical points above are connected by a solution that goes from the point with the largest number of positive eigenvalues to the point with the largest number of negative eigenvalues. Sketch \( U(x - ct) \) and \( W(x - ct) \) for \( t = 0, 1, 2 \).

(h) Compute the minimum wave speed for which the critical points have only real eigenvalues. At the minimum wave speed, how many years would it take for the Numic population to overtake the great basin? (Use a good 1-dimensional approximation for the distance across the great basin.)