1. Solve the following constant-coefficient difference equations.

(a) 
\[ x_{k+1} - x_k - 6x_{k-1} = 0 \]
\[ x_0 = 3 \]
\[ x_1 = 4 \]

(b) 
\[ x_{k+1} + 6x_k + 9x_{k-1} = 0 \]
\[ x_0 = 1 \]
\[ x_1 = 0 \]

(c) 
\[ x_{k+1} + 9x_{k-1} = 0 \]
\[ x_0 = 3 \]
\[ x_1 = 2 \]

(d) 
\[ x_{k+1} - 2x_k + 2x_{k-1} = 0 \]
\[ x_0 = 2 \]
\[ x_1 = 0 \]

(e) 
\[ x_{k+1} + 2x_k + 4x_{k-1} = 0 \]
\[ x_0 = 0 \]
\[ x_1 = 1 \]

2. 42) 2,3,4,5,6,7,9.

3. This continues 42) # 7 above. Recall that the linearization from part a) is

\[ \frac{dx}{dt} = -ax(t - t_d). \]

(a) The Lambert-W function is defined as \( W(x) = f^{-1}(x) \), where \( f(x) = xe^x \). Show that if \( a < 0 \) then \( x = e^{rt} \) is a solution when \( r = \frac{1}{td} W(-at_d) \)

(b) Show that if \( a > 0 \) then there are two solutions of the form \( e^{rt} \) when \( 0 < t_d < \frac{1}{ae} \).

(Hint: There are lots of ways to do this. Draw the picture of \( r = -ae^{-rt_d} \) when there are two solutions. Now subtract \( r \) and \(-ae^{-rt} \). There are then two solutions when \( r + ae^{-rt_d} \) crosses below the \( r \) axis. Solve for the value of \( r \) that gives the minimum, and then set the minimum to zero.)
(c) This means when \( a > 0 \) and \( at_d > \frac{1}{e} \), there are no real solutions for \( r \), but there are complex solutions. Show that if \( r = \alpha + \beta i \) solves \( r = -ae^{-rt_d} \) then so does \( r = \alpha - \beta i \). Use this to conclude that there are oscillatory solutions to \( \frac{dx}{dt} = -ax(t-t_d) \) of the form \( x = e^{\alpha t} \cos(\beta t) \). (Use Euler’s Identity!)

(d) When \( \alpha > 0 \), the oscillatory solutions \( e^{\alpha t} \cos(\beta t) \) become unstable. Show this happens when \( at_d > \frac{\pi}{2} \). (Hint: Use Euler’s Identity and set \( \alpha = 0 \) to solve \( r = -ae^{-t_dr} \).)

(e) It is a (nonobvious) theorem that a critical point of a nonlinear delay DE is stable if and only if the solutions of the form \( x(t) = e^{rt} \) of the linearization at the critical point satisfy \( \lim_{t \to \infty} x(t) = 0 \). We have been discussing the linearization of the logistic equation around the equilibrium \( N = \frac{a}{b} \). For what values of \( t_d \) are there stable oscillations around the critical point? For what values of \( t_d \) do the oscillations become unstable?

(f) How does this compare with the answer for the discrete logistic equation with delay? (Section 42... look for the values of \( a\Delta t \) that give convergent oscillation and unstable oscillation, and remember that \( t_d = \Delta t \) there.)

4. 
\[
\begin{align*}
\frac{dx}{dt} &= x(9 - 3y) \\
\frac{dy}{dt} &= y(10x - 5)
\end{align*}
\]

(a) Find the nullclines and direction arrows.

(b) Sketch the general solution.

(c) Sketch \( x(t) \) and \( y(t) \) if \( x(0) = 1, y(0) = 3 \).

5. 
\[
\begin{align*}
\frac{dx}{dt} &= x(5 - x - 3y) \\
\frac{dy}{dt} &= y(3x - 6)
\end{align*}
\]

(a) Find the nullclines and direction arrows.

(b) Show that the critical point with \( x_c, y_c > 0 \) is a spiral in, by finding the eigenvalues of the Jacobian.

(c) Sketch the general solution.

(d) Sketch \( x(t) \) and \( y(t) \) if \( x(0) = 1, y(0) = 2 \).

6. 50) 4,5,7,12,15