HW 4 Solutions

1. Note: if \( r = Re^{i\theta} \), then \( r^m = (Re^{i\theta})^m = R^m e^{im\theta} = R^m (\cos (m\theta) + i \sin (m\theta)) \).

a) \[
\begin{align*}
\begin{cases}
    x_{k+1} - x_k - 6x_{k-1} &= 0 \\
    x_0 &= 3 \\
    x_1 &= 4
\end{cases}
\end{align*}
\]

\[ x_k = a(-2)^k + b(3)^k \]
\[ 3 = a + b, \quad 4 = -2a + 3b, \]
\[ 5b = 10, \text{ or: } b = 2, a = 1. \]

Thus \( x_k = (-2)^k + 2(3)^k \).

b) \[
\begin{align*}
\begin{cases}
    x_{k+1} + 6x_k + 9x_{k-1} &= 0 \\
    x_0 &= 1 \\
    x_1 &= 0
\end{cases}
\end{align*}
\]

\[ r^2 + 6r + 9 = 0 \]
\[ (r+3)^2 = 0 \]

Thus \( x_k = a(3)^k + b(3)^k \).

\[ 1 = a, \quad 0 = -3a - 3b. \text{ So } b = -1. \]

Thus \( x_k = (3)^k - k(3)^k \).

c) \[
\begin{align*}
\begin{cases}
    x_{k+1} + 9x_{k-1} &= 0 \\
    x_0 &= 3 \\
    x_1 &= 0
\end{cases}
\end{align*}
\]

\[ r^2 + 9 = 0 \]
\[ (r - 3i)(r + 3i) = 0 \]

So \( x_k = a(3i)^k + b(-3i)^k \)

\[ = a(3^k) e^{i\pi k/2} + b(3^k) e^{-i\pi k/2} \]

\[ = (3^k) \left[ a \cos \left( \frac{\pi k}{2} \right) + i \sin \left( \frac{\pi k}{2} \right) \right] + k a \cos \left( \frac{\pi k}{2} \right) - b \sin \left( \frac{\pi k}{2} \right) \]

\[ = (3^k) \left[ (a + bk) \cos \left( \frac{\pi k}{2} \right) + i (a - b) \sin \left( \frac{\pi k}{2} \right) \right] \]

where \( d = a + b \)
\[ \beta = i(a - b). \]
\[
\begin{align*}
&d) \quad \begin{cases}
x_{k+1} - 2x_k + 2x_{k-1} = 0 \\
x_0 = 2 \\
x_1 = 0
\end{cases} \\
\Rightarrow \quad \frac{r^2 - 2r + 2 = 0}{(r-1)^2 + 1}, \text{ so } r = 1 \pm i = \sqrt{2} e^{ \pm i \pi / 4}
\end{align*}
\]

\[X_k = 2^{k/2} \left( a \cos \left( \frac{k \pi}{4} \right) + b \sin \left( \frac{k \pi}{4} \right) \right)\]

\[
\begin{align*}
z &= a \\
o &= 2^{1/2} \left( \frac{a}{2} + \frac{b}{2} \right), \text{ thus } X_k = 2^{k/2} \left( 2 \cos \left( \frac{k \pi}{4} \right) - 2 \sin \left( \frac{k \pi}{4} \right) \right).
\end{align*}
\]

\[e) \quad \begin{cases}
x_{k+1} + 2x_k + 4x_{k-1} = 0 \\
x_0 = 0 \\
x_1 = 1
\end{cases} \\
\Rightarrow \quad \frac{r^2 + 2r + 4 = 0}{(r+1)^2 + 3}, \text{ so } r = -1 \pm \sqrt{3} i = 2 e^{ \pm 2i \pi / 3}
\]

\[X_k = 2^k \left( a \cos \left( \frac{2k \pi}{3} \right) + b \sin \left( \frac{2k \pi}{3} \right) \right)\]

\[
\begin{align*}
o &= a \\
o &= 2^{1/2} \left( \frac{a}{2} + \frac{b \sqrt{3}}{2} \right), \text{ thus } X_k = 2^k \left( \frac{1}{2} \sin \left( \frac{2k \pi}{3} \right) \right) = 2^k \left( \frac{\sin \left( \frac{2k \pi}{3} \right)}{2} \right)
\end{align*}
\]

2. 42.2: a) \[N_{k+1} = N_k + \Delta t N_{k-1} \begin{bmatrix} a - bN_k \end{bmatrix}. \text{ If } N_k \equiv N_E, \text{ then }\]

\[o = \Delta t N_E \begin{bmatrix} a - bN_E \end{bmatrix}, \text{ so } N_E = 0, \frac{a}{b} \text{ are the two equilibria,}\]

b) \[N_m = N_E + \varepsilon y_m, \text{ where } N_E = \frac{a}{b} \neq 0 \text{ and } |\varepsilon| \text{ is small:}\]

\[\varepsilon \begin{bmatrix} y_{k+1} - y_k \end{bmatrix} = \Delta t \begin{bmatrix} N_E + \varepsilon y_{k-1} \end{bmatrix} \begin{bmatrix} a - bN_E \end{bmatrix} - \varepsilon^2 \begin{bmatrix} y_{k-1} \end{bmatrix}^2, \text{ so }\]

\[y_{k+1} - y_k \approx -a \Delta t y_{k-1}. \]

c) \[a \Delta t = \frac{1}{2}: \quad y_{k+1} - y_k + \frac{1}{2} y_{k-1} = 0 \Rightarrow \frac{r^2 - r + 1}{2} = 0\]

\[\frac{1}{2} \left( \frac{\sqrt{5} - 1}{2} + i \frac{\sqrt{5} - 1}{2} \right) = \frac{1}{2} \pm \frac{1}{2} \frac{\sqrt{3}}{5} e^{ \pm i \pi / 4}\]

\[y_k = 2^{-k/2} \begin{bmatrix} a \cos \left( \frac{k \pi}{4} \right) + b \sin \left( \frac{k \pi}{4} \right) \end{bmatrix}, \text{ so the displacement } y_k \text{ from the equilibrium } \frac{a}{b} \text{ shrinks by a factor of } \sqrt{2} \text{ with } k \text{ and oscillates with a period of } 8 \text{ with } k \text{ (i.e., } y_{k+8} = 2^{-8} y_k).} \]
42.2 (cont.)

\[ N_k = 0 + \varepsilon y_k \Rightarrow y_{k+1} - y_k = \Delta t y_{k-1}(a) \]

\[ r^2 - r = a \Delta t > 0, \text{ so } (r - \frac{1}{2})^2 - \frac{1}{4} = a \Delta t, \text{ or} \]

\[ r = \frac{1 \pm \sqrt{1 + 4a \Delta t}}{2} \]

The dominant root, and thus the root governing the behavior of the displacement from \( N_0 = 0 \), is \( r_1 = \frac{1 + \sqrt{1 + 4a \Delta t}}{2} > 1 \), so \( y_k \) will grow with time.

\[ y_k = a r_1^k + b r_2^k \]

so the asymptotic growth rate is:

\[ \lim_{k \to \infty} \frac{y_{k+1} - y_k}{\Delta t y_k} = \lim_{k \to \infty} \frac{a r_1^k + b r_2^k}{\Delta t (a r_1^k + b r_2^k)} = \lim_{k \to \infty} \frac{a r_1^k - a}{\Delta t} = \frac{\frac{1}{2} - \frac{1}{2}}{\Delta t} = -\frac{1}{2a} \]

Note that

\[ \lim_{\Delta t \to 0} \frac{r_1 - 1}{\Delta t} = \lim_{\Delta t \to 0} \frac{4a}{4 \sqrt{1 + 4a \Delta t}} = a, \text{ so } y_{k+1} \approx (1 + a \Delta t) y_k \text{ if } \Delta t \text{ is small.} \]

Since \( N_k = 3 y_k \approx (1 + a \Delta t)3 y_{k-1} = (1 + a \Delta t) N_{k-1} \) and

\[ N_k = N(t_0 + k \Delta t), \text{ we have } N(t) \approx (1 + a \Delta t)^k N_0 = \left( 1 + a \Delta t \right)^{t-t_0} N_0 \]

\[ \approx N_0 e^{\frac{a(t-t_0)}{\Delta t}} \text{ if } \Delta t \text{ is small.} \]

Rigorously:

\[ \lim_{\Delta t \to 0} (1 + a \Delta t)^{t-t_0} = e^{\frac{a(t-t_0)}{\Delta t}} \]

42.3 (1) The perturbation obeys \( y_{m+1} - y_m = -a \Delta t y_m \), so \( y_{m+1} = (1 - a \Delta t) y_m \), and we get that \( y_m \) grows if \( |1 - a \Delta t| > 1 \), shrinks if \( |1 - a \Delta t| < 1 \), and oscillates (either growing or shrinking) if \( (1 - a \Delta t) < 0 \). Thus \( a \Delta t > 2 \) indicates growth of \( y_m \) (unstable equilibrium @ \( N_E = \frac{a}{b} \)), \( a \Delta t < 2 \) indicates decay of \( y_m \) (stable equilibrium @ \( N_E = \frac{a}{b} \)) and the displacement \( y_m \) oscillates with \( m \) if \( a \Delta t > 1 \). Negative populations will thus result if \( a \Delta t > 2 \). Comparing these results to those of section 42, we see that introducing more of a delay, using model (2) causes greater instability. In model (2), \( a \Delta t \) needs to be smaller than \( 1 \) to achieve stability.
42.4 a) $N_k = N_E : N_E = \frac{a}{b}$, as demonstrated before.
b) $N_{k+1} - N_k = \Delta t [N_{k-1} (a - b N_{k-1})]$, so if $N_k = N_E + \varepsilon y_k$, where $N_E = \frac{a}{b}$, then:

$$\varepsilon y_{k+1} - \varepsilon y_k = \Delta t (\frac{a}{b} + \varepsilon y_{k+1}) (1 - b \varepsilon y_k),$$
so $y_{k+1} - y_k \approx -\Delta t y_{k-1}$, and thus for small $|\varepsilon|$, $y_k$ grows as $|1 + \frac{1 - a \Delta t}{2}|$ (even if $(1 - a \Delta t < 0)$) so $N_E = \frac{a}{b}$ is stable iff $1 - a \Delta t > 0$, or $\Delta t < 1$. Decaying oscillation if $\frac{1}{2} < a \Delta t < 1$.

c) Same stability results.

42.5 a) $N_k = N_E : N_E = \frac{a}{b}$ is the nonzero equilibrium population, as derived before.
b) $N_{k+1} - N_k = \Delta t [N_k (a - b N_k)]$, so if $N_k = \frac{a}{b} + \varepsilon y_k$:

$$y_{k+1} - y_k \approx -\Delta t y_{k-2}.$$ Characteristic equation:

$$r^3 - r^2 + \frac{a \Delta t}{b}$$

$$r^3 - r^2 = r^2 (r-1),$$ so:

$$r^2 - r + \frac{a \Delta t}{b} = 0.$$ 

In the case of 3 distinct roots $-(\Delta t - 1) + \sqrt{(\Delta t - 1)^2 - 4 a \Delta t}$, assuming $\Delta t < 2$ and all real roots are real/distinct, they will lie in $(-1, 1)$, and so $N_E = \frac{a}{b}$ is a stable equilibrium in this case.
42.6 \( N_{k+1} - N_k = R_0 \Delta t \cdot N_{k-2} \). Characteristic eqn: \( r^3 - r^2 = R_0 \Delta t \).

Using \( R_0 \Delta t = \varepsilon \), \( 0 < \varepsilon \ll 1 \).

For the new roots to be perturbations from \( r = 0, 1 \). Try \( r = \varepsilon p \) and \( r = 1 + \varepsilon q \). Then for \( r = \varepsilon p \): \( \varepsilon^3 p^3 - \varepsilon^2 p^2 = \varepsilon \), so \( \varepsilon (\varepsilon p^3 - p^2) = 1 \), so getting rid of the \( O(\varepsilon) \) term, \( -\varepsilon^2 = 1 \), so \( p = \pm i/\sqrt{\varepsilon} \). For \( r = 1 + \varepsilon q \): \( r^3 - r^2 = \varepsilon = r^2 (r - 1) \), so \( \varepsilon = (1 + \varepsilon q)^2 \) or \( (1 + \varepsilon q)^2 = 1 \). Ignoring the \( O(\varepsilon) \) terms gives \( q = 1 \). Note that since \( p = O(1/\sqrt{\varepsilon}) \), the "\( O(\varepsilon^2) \)" term in that calculation ended up really being \( O(1/\varepsilon) \), which is still ok.

So the approximate general solution is: \( N_k = N(t_0 + k\Delta t) = (R_0 \Delta t)^{k/2} \left[ a \cos \left( \frac{\pi k}{2} \right) + b \sin \left( \frac{\pi k}{2} \right) + c(1 + R_0 \Delta t) \right] \).

and since \( |1 + R_0 \Delta t| > 1 \), the solution grows roughly exponentially with time.

42.7 a) \( N(t) = N_E + \varepsilon y(t) \): \( \varepsilon y'(t) = (N_E + \varepsilon y(t)) \left[ a - b N_E - b\varepsilon y(t - \tau_m) \right] \)

\[ \begin{align*}
N(E) & = N_E \left[ a - b N_E \right] + \varepsilon y(t) \left[ a - b N_E \right] \\
& - b \varepsilon y(t) y(t - \tau_m) - N_E b \varepsilon y(t - \tau_m).
\end{align*} \]

Dividing by \( \varepsilon \) and omitting the \( O(\varepsilon) \) term:

\( y'(t) = a y(t - \tau_m) \) for \( N_E = 0 \), and \( y'(t) = -a y(t - \tau_m) \) for \( N_E = \frac{a}{b} \).

b) For \( N_E = \frac{a}{b} \), try \( y(t) = e^{rt} \). \( r e^{rt} = -a e^{(t - \tau_m)} = a e^{-\tau_m} r t \), so \( r = -a e^{-\tau_m} \).

c) \( r = -a e^{-\tau_m} \). So these exist if \( \tau_m \) is small but not if \( \tau_m \) is large.
\[ y_{n+1} - y_n = -d y_{n-1}; \text{ characteristic roots: } r^2 - r = -d, \text{ so} \]
\[ r = \frac{1 \pm \sqrt{1 - 4d}}{2}, \text{ and since } 1 - 4d < -3, \]
and \[ \left| \frac{1 \pm \sqrt{1 - 4d}}{2} \right| = 1, \]
y will grow according to \( \left( \frac{1 \pm \sqrt{1 - 4d}}{2} \right)^n \). Thus we get growing oscillations, and the period of this oscillation is
\[ \frac{2\pi}{\arctan \left( \frac{\sqrt{1 - 4d}}{2} \right)} = \frac{2\pi}{\arctan \left( \frac{\sqrt{1 - 4d}}{2} \right)} . \]
This period is smallest when \( e \to \infty \), in which case it is \( \frac{2\pi}{\pi/2} = 4 \).

3. a) Trying \( x = e^{rt} \) yields:
\[ re^{rt} = x' = -ax(t - td) = -ae^{-rtd} e^{rt} \]
\[ r = -ae^{-rtd}, \text{ so} \]
\[ \frac{(rt_d)e^{-rt_d}}{f(rt_d)} = -at_d . \] This occurs only if \( f(rt_d) = -at_d \), or equivalently, \( rt_d = f^{-1}(-at_d) = W(-at_d) \).

Thus \( r = \frac{1}{td} W(-at_d) \). Since \( f(x) = 0 \) when \( x = -1 \), \( f \) is only one-to-one on \( (-\infty, -1] \) and \( [1, \infty) \) separately, and since \( f(-1) = \frac{1}{e} \), we must use \( f: [-1, \infty) \to [-\frac{1}{e}, \infty) \) to get the largest codomain. Then \( \hat{W} = f^{-1} \): \( [-\frac{1}{e}, \infty) \to [-1, \infty) \), so we require \( -at_d \geq -\frac{1}{e} \), or
\[ a \leq \frac{1}{e \cdot td}, \text{ which is certainly satisfied if } a < 0. \]

b) \( f \) is decreasing on \( (-\infty, -1] \) and increasing on \( [-1, \infty) \), and is continuous.
Since \( \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x e^x = \lim_{x \to -\infty} \frac{x}{\frac{e^x}{x}} = \lim_{x \to -\infty} \frac{1}{e^x} = 0 \), \( f(-1) = \frac{1}{e} \),
and \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} x e^x = \infty \), by the Intermediate Value Theorem (IVT), \( f(x) = y \) has exactly 2 solutions if \( -\frac{1}{e} < y < 0 \), and exactly 1 solution if \( y = -\frac{1}{e} \) or \( y \geq 0 \). Thus there are 2 thus when \( f(rt_d) = -at_d e^{-\frac{1}{e}} \), or \( \text{otherwise}. \)
3. (cont.) c) Note that there are no solutions to \( f(rt_d) = -at_d \) if \(-at_d < -\frac{\epsilon}{e}\), which happens when \( at_d > \frac{1}{e} \). It's possible to show, using complex analysis & Picard's Big Theorem, that there are (infinitely many) complex solutions \( r \) to \( f(rt_d) = -at_d \), as long as \( a \neq 0 \neq t_d \). If \( r = x + iy \) solves \( r = -ae^{-at_d} \), then

\[
(x+i\beta)t_d = -(x+i\beta)e^{-at_d} = -(x+i\beta)e^{-at_d}(\cos(\beta t_d) + i\sin(\beta t_d)) = -ae^{-at_d}(\cos(\beta t_d) + i\sin(\beta t_d)) = -ae^{-at_d}e^{i\beta t_d} = -ae^{-at_d}e^{2\pi i \beta t_d}
\]

and so \( r = x - iy \) also satisfies \( r = -ae^{-rt_d} \) as a result (assuming \( x, y \in \mathbb{R} \)). Thus, using the trial solution \( x = e^{-rt_d} \) discussed in part (a), we get solutions of the form

\[
x = c_1 e^{(x+iy)t_d} + c_2 e^{(x-iy)t_d} = e^{at_d} \left[ c_1 (\cos(\beta t_d) + i\sin(\beta t_d)) + c_2 (\cos(\beta t_d) - i\sin(\beta t_d)) \right] = e^{at_d} \left[ (c_1 + c_2) \cos(\beta t_d) + (ic_1 - ic_2) \sin(\beta t_d) \right], \text{ where } c_1, c_2 \in \mathbb{C}.
\]

Since \( c_1 \) & \( c_2 \) are arbitrary, we may take \( c_1 = \frac{1}{2} = c_2 \) to get

\[
x = e^{at_d} \cos(\beta t_d) \text{ as a solution.}
\]

d) Observation: as \( t_d \to \infty \), instability 1. So let's look for the "threshold" \( t_d \) value that produces \( a = 0 \), which is when equilibria start to become unstable. \( r = 0 + i\beta \), \( r = -ae^{-rt_d} \) means:

\[
i\beta = -ae^{-at_d} = -a(\cos(\beta t_d) - i\sin(\beta t_d)).
\]

The real part, \(-acos(\beta t_d)\) must be 0, so take \( \beta t_d = \frac{\pi}{2} + \pi n \), \( n \in \mathbb{Z} \). Then \( \beta = a\sin(\frac{\pi}{2} + \pi n) = \pm \alpha \), and thus \( at_d = \frac{\pi}{2} + \pi n \). The first value of \( t_d \) that produces instability is given by \( at_d = \frac{\pi}{2} \).
3. (cont.)

e) From part b, we know that \(0 < t_d < \frac{1}{ae}\) produces solutions that don't oscillate, and \(\frac{1}{ae} < t_d\) causes oscillations.

From part d, we know that \(0 < t_d < \frac{\pi}{2a}\) produces stable solutions and \(\frac{\pi}{2a} < t_d\) produces instability. Thus if \(0 < t_d < \frac{1}{ae}\), then the equilibrium is an attracting node (stable, non-oscillating).

If \(\frac{1}{ae} < t_d < \frac{\pi}{2a}\), then the equilibrium is an inward-pointing spiral (stable, oscillating). If \(\frac{\pi}{2a} < t_d\), then the equilibrium is an outward-pointing spiral (unstable, oscillating).

f) There, \(0 < \Delta t < \frac{1}{4a}\) produces attracting nodes while \(\frac{1}{4a} < \Delta t < \frac{1}{a}\) produces inward spirals and \(\frac{1}{a} < \Delta t\) produces unstable outward spirals, so since \(\frac{1}{4} < \frac{1}{e}\) and \(1 < \frac{\pi}{2}\), here the analogous regions for \(\Delta t\) (\(t_d\) now) allow for more flexibility in choosing \(t_d\).

4.

\[
\begin{align*}
x' &= x(9 - 3y) \\
y' &= y(10x - 5)
\end{align*}
\]

a) \(x' = 0 \iff (x = 0 \text{ or } y = 3)
\]
\(y' = 0 \iff (y = 0 \text{ or } x = \frac{1}{2})\)

Note: not necessary to draw the relative lengths of direction arrows...

Also, find directions in regions 1 & 2 by using a test pair of \((x, y)\) in that region.

b) by the graph from (a).

Notes: by the picture alone, it is too difficult to distinguish between centers & inward/outward spirals.

c) by part (c).
5. \[ x' = x(5 - x - 3y) \\
y' = y(3x - 6) \]
a) \[ x' = 0 \iff \left\{ \begin{array}{l} x = 0 \\
y = \frac{5}{3} - \frac{x}{3} \end{array} \right. \]
\[ y' = 0 \iff \left\{ \begin{array}{l} y = 0 \\
x = 2 \end{array} \right. \]

b) \((x_c, y_c) = (2, 1)\). If \[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} F(x, y) \\ G(x, y) \end{bmatrix}, \]
then by the
equilibrium point \((x_c, y_c)\), we have \[ J = \text{Joc} \left( \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \right)_{(x_c, y_c)} \].
Thus \[ J = \begin{bmatrix} 10 - 3y & -3x \\ 3y & 3x - 6 \end{bmatrix} \]
\[(2, 1) \begin{bmatrix} -2 & -6 \\ 3 & 0 \end{bmatrix}, \]
which has eigenvalues:
\[ \det(J - \lambda I) = (2 - \lambda)(6 - \lambda) + 18 \implies \lambda^2 + 2\lambda + 18 = 0, \]
\[ \lambda = 1 \pm \sqrt{11 - 12} = 1 \pm \sqrt{11 - 12}, \]
which means \[ \text{Re}(r_1) = -1 = \text{Re}(r_2), \]
so since both eigenvalues have negative real parts, and \(r_1, r_2\)
equal complex conjugates of each other, we get that the
critical point \((x_c, y_c) = (2, 1)\) is an inward spiral (stable).

\(\text{c) diagram}\)
\(\text{d) diagram}\)
\begin{align*}
F' &= F(a-bF-cS), \quad F = \frac{k}{\lambda} + x, \quad S = \frac{a}{c} - \frac{bk}{c^2} + y \\
S' &= S(-k + \lambda F),
\end{align*}

(1) Using \( u(F,S) = F(a-bF-cS) \) we get
\[
\begin{align*}
V(F,S) &= S(-k + \lambda F), \\
x' &= v_F(F_E,S_E)x + u_S(F_E,S_E)y \\
y' &= v_F(F_E,S_E)x + v_S(F_E,S_E)y.
\end{align*}
\]
Since \( u_F = a - cS - 2bF, \quad u_S = -cF \)
\[
\begin{align*}
v_F &= \lambda S, \quad v_S = -k + \lambda F,
\end{align*}
\]
we get
\[
\begin{align*}
x' &= \left[a - c\left(\frac{a}{c} - \frac{bk}{c^2}\right) - 2b\left(\frac{k}{\lambda}\right)\right]x + \left[-c\left(\frac{k}{\lambda}\right)\right]y \\
y' &= \left[\lambda\left(\frac{a}{c} - \frac{bk}{c^2}\right)\right]x + \left[-k + \lambda \left(\frac{k}{\lambda}\right)\right]y, \quad \text{or:}
\end{align*}
\]
\[
\begin{align*}
x' &= (-bk)x + (-ck)y \\
y' &= (a^2 - bk)c x.
\end{align*}
\]

(2) Using just \( F = \frac{k}{\lambda} + x, \quad S = \frac{a}{c} - \frac{bk}{c^2} + y \) and omitting the small \( O(x^2, y^2, xy) \) terms yields: (using \( u(F,S) = 0 \)
\[
\begin{align*}
x' &= (\frac{k}{\lambda} + x)(-bx - cy) \\
y' &= y(-k + \lambda (\frac{k}{\lambda} + x)) + (\frac{a}{c} - \frac{bk}{c^2})(\lambda x)
\end{align*}
\]
\[
\begin{align*}
x' &\approx (-bk)x + (-ck)y \\
y' &\approx (\frac{a^2}{c} - bk)c x, \quad \text{giving the same result.}
\end{align*}
\]
\[ S_i'' = \left( \frac{a - \frac{bk}{c \lambda}}{c \lambda} \right) \lambda F_i' = \left( \frac{a - \frac{bk}{c \lambda}}{c \lambda} \right) \lambda \left[ \frac{k}{\lambda} \left( -bF_i - cS_i \right) \right] \]
\[ = \left( \frac{a - \frac{bk}{c \lambda}}{c \lambda} \right) \lambda \left( -\frac{a S_i}{\lambda} \right) + \left( \frac{a - \frac{bk}{c \lambda}}{c \lambda} \right) \lambda \left( \frac{b}{\lambda} \right) \left( -b \left( \frac{S_i'}{a - \frac{bk}{c \lambda}} \right) \right) \]
\[ = \left( \frac{bk^2}{\lambda^2} - \frac{ak}{\lambda} \right) S_i' - \left( \frac{kb}{\lambda} \right) S_i' \quad \text{Thus } S_i'' + \alpha S_i' + \beta S_i = 0 \]
needs to be solved. This can be accomplished by using the
trial solution \( S_i'(t) = e^{rt} \) and solving for \( r \). This
results in:
\[ r = \frac{-a \pm \sqrt{a^2 - 4\beta}}{2} = \frac{-\left( \frac{kb}{\lambda} \right) \pm \sqrt{\left( \frac{kb}{\lambda} \right)^2 + 4 \left( \frac{bk^2}{\lambda^2} - \frac{ak}{\lambda} \right)}}{2} \]
Assuming \( k > 0, a, b > 0 \), this means we get two distinct, negative
real roots if \( -\left( \frac{kb}{\lambda} \right)^2 < 4 \left( \frac{bk^2}{\lambda^2} - \frac{ak}{\lambda} \right) < 0 \), in which case
the equilibrium is stable. Note: If \( S_i \to 0 \), then \( S_i' \to 0 \), so \( F_i \to 0 \). Thus stability of shark
population implies stability of fish population.
We get complex conjugate roots with negative real part if
\[ 4 \left( \frac{bk^2}{\lambda^2} - \frac{ak}{\lambda} \right) < -\left( \frac{kb}{\lambda} \right)^2 \], so here the equilibrium
would be stable and oscillating. We get also that
\[ 4 \left( \frac{bk^2}{\lambda^2} - \frac{ak}{\lambda} \right) > 0 \] causes one of the roots to be positive,
so the equilibrium is unstable in this case.
in this case, \( 4 \left( \frac{bk^2}{\lambda^2} - \frac{ak}{\lambda} \right) = 0 \) : neutrally stable.
\( \left( \frac{kb}{\lambda} \right)^2 + 4 \left( \frac{bk^2}{\lambda^2} - \frac{ak}{\lambda} \right) = 0 \) : attracting node (stable).
Linearizing/analyzing the local behavior of equation 50.1 around the equilibrium \( F_E = \frac{a}{l} \), \( S_E = \frac{a}{c} - \frac{b F}{c} \), using \( F = F_E + x, \ S = S_E + y \), yields:

\[
\begin{align*}
  x' &= -c_1 x - c_2 y, \\
  y' &= c_3 x,
\end{align*}
\]

where \( c_1 = \frac{kb}{\lambda} > 0 \), \( c_2 = \frac{c k}{\lambda} > 0 \), and \( c_3 = \left( \frac{a}{c} - \frac{b F}{c} \right) \lambda < 0 \).

Nullclines:

\[
\begin{align*}
  x' &= 0 & \iff & y = -\frac{c_1}{c_2} x, \\
  y' &= 0 & \iff & x = 0
\end{align*}
\]

This equilibrium is a saddle point \((F_E, S_E)\), described above. But! Since \( S_E < 0 \), and this cannot happen in reality, we omit this equilibrium point from consideration.

Now, looking @ \((F, S)\) instead of \((x, y)\):

\[
\begin{align*}
  F' &= F(a - bF - cS) = 0 \iff (F = 0 \text{ or } S = \frac{a - b F}{c}) \\
  S' &= S(c + \lambda F) = 0 \iff (S = 0 \text{ or } F = \frac{b}{\lambda})
\end{align*}
\]

and the only (possibly) equilibrium point is \((F_E, S_E, F(0))\), which is unstable and steers \((F, S)\) towards \((\frac{a}{b}, 0)\).

By the slope field, all solutions look like the one pictured, and eventually, the sharks will die off and fish will approach \((\frac{a}{b}, 0)\).
50.12 \[ F' = F(a-bF-cS) \]
\[ S' = S(-k+\lambda F-\sigma S) \]

The only difference is the introduction of the \(-\sigma S^2\) term, indicating a carrying capacity for the sharks has been added.

The non-zero equilibrium populations are given by the intersection of the two lines \(a-bF-cS=0\) and \(-k+\lambda F-\sigma S=0\). We can roughly obtain general solutions via nullclines for \(F\) & \(S\): (assuming \(S<\frac{a}{c}\), else similar to last problem)

\[ S = \frac{k+\lambda F}{b} \]

But this only tells us that \((F_E, S_E)\), the non-zero equilibrium (the others are \((F, S)=(\frac{a}{c}, 0)\) and \((F, S)=(0, 0)\)), corresponds to either a spiral or a center. To find out which, we linearize: Set \(u(F, S) = F(a-bF-cS)\)
\[ V(F, S) = S(-k+\lambda F-\sigma S) \]

Solving \(V(F, S) = 0\) yields:
\[ \begin{align*}
  x' &= u_x(F_E, S_E)x + u_y(F_E, S_E)y = (a-cS_E-2bF_E)x - (cF_E)y \\
  y' &= V_x(F_E, S_E)x + V_y(F_E, S_E)y = (\lambda S_E)x + (-k+\lambda F_E-2\sigma S_E)y \\
  &= (\lambda S_E)x - (\sigma S_E)y \\
\end{align*} \]

So \((x', y') = \mathbf{J}(\mathbf{x})\), where \(\mathbf{J} = \begin{pmatrix} -bF_E - cF_E & cF_E \\ \lambda S_E & -\sigma S_E \end{pmatrix}\). Eigenvalues:
\[ (-bF_E - \lambda S_E)(-\sigma S_E) + c\lambda S_E F_E = 0 \]
\[ \leftrightarrow \lambda^2 + (bF_E + \sigma S_E)\lambda + (b\sigma + c\lambda)S_E F_E = 0 \]
\[ \leftrightarrow \lambda = \frac{-bF_E - \sigma S_E \pm \sqrt{(bF_E + \sigma S_E)^2 - 4(b\sigma + c\lambda)S_E F_E}}{2} \]

Since \(-bF_E - \sigma S_E < 0\) and \(4(b\sigma + c\lambda)S_E F_E > 0\), we know that we will get either two distinct negative roots, repeated negative roots, or complex conjugate roots with negative real part. In all of these scenarios (by the nullclines, it is in fact the last case, though it'd be more work to check that), the equilibrium \((F_E, S_E)\) is stable.
\[
\frac{dF_i}{ds_i} = \frac{dF_i/dt}{ds_i/dt} = \frac{k (bF_i - CS_i)}{(c - \frac{bk}{c_0}) F_i}
\]

\[
\frac{dF_i}{ds_i} = 0 \text{ when } F_i = \frac{c}{b} S_i, \quad \text{and } \frac{dF_i}{ds_i} = \infty \text{ when } F_i = 0.
\]

We can summarize the sign of \(\frac{dF_i}{ds_i}\) using a table:

<table>
<thead>
<tr>
<th>(F_i)</th>
<th>(\frac{dF_i}{ds_i})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_i &lt; \frac{c}{b} S_i)</td>
<td>neg</td>
</tr>
<tr>
<td>(F_i = \frac{c}{b} S_i)</td>
<td>0</td>
</tr>
<tr>
<td>(F_i &gt; \frac{c}{b} S_i)</td>
<td>pos</td>
</tr>
</tbody>
</table>

\(F_i < 0\): \(\text{vertical}\) (vertical)
\(F_i = 0\): \(\text{Indeterminate}\) (vertical)
\(F_i > 0\): \(\text{pos}\)

Note: at the top of one of its spiral-like trajectory,
\(F_i > 0, S_i = 0 \Rightarrow \frac{dF_i}{dt} < 0\),
\(\frac{dF_i}{ds_i} < 0\), and thus the spiral moves clockwise.

These are the two qualitatively distinct possible trajectories.

inward spiral, clockwise

outward spiral, clockwise