77.1 \[ \frac{dx_1}{dt} = \frac{p_1 u_1 - p_0 u_0}{p_1 - p_0} = \frac{p_1 u_m (1 - \frac{p_1}{p_m}) - p_0 u_m (1 - \frac{p_0}{p_m})}{p_1 - p_0} = u_m (1 - \frac{p_0 + p_1}{p_m}). \]

Avg of density wave velocities:

\[ \frac{q'(p_1) + q'(p_0)}{2} = \frac{u_m (1 - \frac{2p_1}{p_m}) + u_m (1 - \frac{2p_0}{p_m})}{2} = u_m (1 - \frac{p_0 + p_1}{p_m}). \]

\( \checkmark \) (same)

77.2 \[ \frac{dx_2}{dt} = \frac{p_1 u_1 - p_0 u_0}{p_1 - p_0} = \frac{p_1 u_m (1 - \frac{p_1^2}{p_m^2}) - p_0 u_m (1 - \frac{p_0^2}{p_m^2})}{p_1 - p_0} = u_m (1 - \frac{p_1^2 + p_0^2 + 2p_1 p_0}{p_m^2}). \]

Whereas avg density wave vel.: \[ \frac{q'(p_1) + q'(p_0)}{2} = \frac{u_m (1 - \frac{3p_1^2}{p_m^2}) + u_m (1 - \frac{3p_0^2}{p_m^2})}{2} = u_m (1 - \frac{3p_1^2 + 3p_0^2}{2p_m^2}). \]

77.5 a) When \( \frac{dp}{dx} > 0 \), there's more traffic up ahead, so drivers would slow down (lower \( u \)). Thus \( \gamma > 0 \) would be required.

b) \[ \sigma = p_t + q_x = p_t + \frac{d}{dx} (pu(p, \frac{dp}{dx})) = p_t + \frac{d}{dx} (pu(p) - u(p)) \]

\[ = p_t + p_x u(p) + pu'(p)p_x - u(p)x \]

is the new conservation of mass PDE.

c) \[ u'(p) = \frac{-u_m}{p_m} \], so (b) tells us that

\[ \sigma = p_t + p_x \left[ u_m (1 - \frac{p}{p_m}) - u_m (\frac{p}{p_m}) \right] - u_x p_x \]

so

\[ p_t + p_x u_m (1 - \frac{2p}{p_m}) = u_x p_x. \]
77.6 a) \( p = f, \ p_t = -Vf', \ p_x = f', \ \text{and} \ p_{xx} = f'' \) (all arguments of \( f \) and \( f' \) are derivatives of \( f \)). So Burger's eqn yields:

\[-Vf' + u_-(1 - \frac{2x}{Vt}) f' = \gamma f'' \]

Upon integrating w.r.t. \( x \):

\[ (u_+ - V)f - \frac{u_+}{V} f^2 = Vf' + C, \] or

\[ f' = \frac{u_+ - V}{V} f - \frac{u_+}{V} f^2 + C, \]

where \( C \) represents an arbitrary constant. By manipulating \( C \), we may raise/lower the quadratic on the R.H.S to our liking. Thus we may assume that the quadratic on the R.H.S has two distinct roots, say, \( p_1 \) & \( p_2 \). Phase plane:

Slope field:

\[ \frac{dy}{dx} = \text{stable equilibrium} \]

Physical interpretation: density of cars increases to \( p_2 \) as \( x \to -\infty \) and decreases to \( p_1 \) as \( x \to +\infty \). So for fixed time, density must increase with \( x \).

b) If \( f(x) = f(x, 0) = \begin{cases} p_1 & \text{if } x < a \\ p_2 & \text{if } x > a \end{cases} \) (\( p_1 < p_2 \)), then

\( p(x, t) = f(x-Vt) = \begin{cases} p_1 & x-Vt < a \\ p_2 & x-Vt > a \end{cases} \)

so at time \( t \), \( p(x, t) \) has a discontinuity at \( x = a + Vt \). Thus \( x_5(t) = a + Vt \), and \( \frac{dx_5}{dt} = V \), so the shockwave moves with velocity \( V \), the same velocity with which the density wave traveled.
\[ a) \quad \frac{\partial u}{\partial t} + \frac{3u}{5} \frac{\partial u}{\partial x} = 0 \]

b) Shockwave emanating from \( x=0 \):
\[
\frac{dx}{dt} = \frac{[q]}{[p]} = um \left( 1 - \frac{p_m}{um} \right) p
\]

\( p \) is \( \frac{p_m}{5} \) to the left of shock, \( \frac{3p_m}{5} \) to the right, \( \frac{um}{5} \), so \( x(t) = \frac{um}{5} \frac{t}{5} \)

Thus \( p(x,t) = \begin{cases} 
\frac{um}{5} & \text{if } x < \frac{um}{5} \\
3\frac{um}{5} & \text{if } x > \frac{um}{5}
\end{cases} \)

\[ \frac{dx}{dt} = \frac{3um}{5} \]

\[ c) \quad \text{if } x(0) = -x_0 < 0, \text{ then: } \frac{dx}{dt} = um \left( 1 - \frac{p_m}{um} \right) \text{, so } x(t) = \frac{4um}{5} t - x_0. \]

This initial trajectory of the car intersects the shock when:
\[ \frac{4um}{5} - x_0 = \frac{um}{5}, \text{ at time } t^* = \frac{5}{3} x_0. \]

Then:
\[ x(t) = \frac{5x_0}{3} \]

Putting both parts together yields:
\[ x(t) = \begin{cases} 
\frac{4um}{5} t - x_0 & \text{for } 0 \leq t \leq \frac{5x_0}{3um} \\
\frac{2um}{5} + \frac{5x_0}{3} & \text{for } t > \frac{5x_0}{3um}
\end{cases} \]

d) \( \text{if } x(0) = x_0 > 0, \text{ then: } \frac{dx}{dt} = um \left( 1 - \frac{3p_m}{um} \right) \text{, so } x(t) = \frac{2um}{5} + x_0. \]

The trajectory of the car never intersects the shockwave (for \( t \geq 0 \)), since the only solutions to \( 2\frac{um}{5} + x_0 = \frac{um}{5} \) are negative.

This implies the car always stays ahead of the shock. This makes intuitive sense, since

\( \text{velocity of shockwave } \frac{dx}{dt} \text{ over } [x_0, \frac{5x_0}{3um}] < \text{ (secant slope over } [0, \frac{5x_0}{3um}] \text{)} = \text{ (velocity @ } \frac{5x_0}{3um} \text{), } \text{ and } \text{the car starts ahead of the shock.} \)
78.3 Shockwave: \( \frac{dx}{dt} = Um \left( 1 - \frac{p_0 + p_1}{\rho_m} \right) = Um \left( 1 - \frac{p_m' + 2\rho_m p_m'}{\rho_m} \right) = 0 \)

So since the shockwave starts at \( x_0 = 0 \) (the discontinuity at \( t = 0 \)), \( x_s(t) = 0 \). Thus the density stays the same to the left and right of the shockwave, so the density stays constant through time.

78.10 a) Shockwave: \( \frac{dx}{dt} = \frac{[q]}{[e]} = \frac{p_2 u(p_2) - p_1 u(p_1)}{e_2 - e_1} = Um \left( 1 - \frac{p_1^2 + p_1 p_2 + p_2^2}{e_m^2} \right) \)

\( x_s(t) = \frac{tm}{2} (1 - \frac{p_1^2 + p_1 p_2 + p_2^2}{e_m^2}) \), Thus

\[ \rho(x,t) = \begin{cases} 
\rho_1 & \text{if } x < \frac{tm}{2} \frac{p_1^2 + p_1 p_2 + p_2^2}{e_m^2}, \\
\rho_2 & \text{if } x > \frac{tm}{2} \frac{p_1^2 + p_1 p_2 + p_2^2}{e_m^2}.
\end{cases} \]

b) no shockwave now, but fan! \( q'(p) = Um \left( 1 - \frac{3p_1^2}{e_m^2} \right) \), so

for \( \frac{tm}{2} \left( 1 - \frac{3p_1^2}{e_m^2} \right) < x < \frac{tm}{2} \left( 1 - \frac{3p_2^2}{e_m^2} \right) \), characteristics emanate from \( x_0 = 0 \) and

\( x' = q' = Um \left( 1 - \frac{3p_1^2}{e_m^2} \right) \), so \( x = \frac{tm}{2} \frac{3p_1^2}{e_m^2} \).

(note: \( x' = \frac{dx}{dt}, \frac{q}{p} = \frac{dp}{dt} \)) Isolating \( \xi \) yields:

\[ \rho = \frac{\rho_m}{\sqrt{3}} \frac{\xi - \frac{3p_1^2}{e_m^2}}{\xi - \frac{3p_2^2}{e_m^2}} \]

for \( \frac{tm}{2} \left( 1 - \frac{3p_1^2}{e_m^2} \right) < x < \frac{tm}{2} \left( 1 - \frac{3p_2^2}{e_m^2} \right) \).

Thus:

\[ \rho(x,t) = \begin{cases} 
\rho_1 & \text{for } \frac{tm}{2} \left( 1 - \frac{3p_1^2}{e_m^2} \right) < x < \frac{tm}{2} \left( 1 - \frac{3p_2^2}{e_m^2} \right), \\
\rho_2 & \text{for } \frac{tm}{2} \left( 1 - \frac{3p_2^2}{e_m^2} \right) < x.
\end{cases} \]
b) If $0 < x_0 < L$, then the characteristic emanating from $x_0$ is given by:

$$x' = q = u_m(1 - \frac{2x_0}{L})u_m = u_m(1 - \frac{2x_0}{L})u_m,$$

So

$$x(t) = u_m(1 - \frac{2x_0}{L})t + x_0.$$

Imposing $t = \frac{L}{2u_m}$ into these characteristics yields

$$x(\frac{L}{2u_m}) = \frac{L}{2} - x_0 + x_0 = \frac{L}{2},$$

regardless of $x_0$. So all the characteristics emanating from $[0, L]$ meet at $t = \frac{L}{2u_m}, x = \frac{L}{2}$.

c) The shock wave's motion is affected only by the density directly to its left and right:

$$\frac{dx_s}{dt} = \frac{\rho_m u_m(\rho_s - \rho_0)}{\rho_m - \rho_0} = 0,$$

so $x_s(t) = \frac{L}{2}$, which points directly up.

d) Before shock ($t < \frac{L}{2u_m}$): For $x < u_m t$, $p = 0$. For $x > L - u_m t$, $p = \rho_m u_m$.

For $u_m t < x < L - u_m t$, $p$ is constant $p(x_0, 0)$ along the characteristics $x = u_m t (1 - \frac{2x_0}{L}) + x_0$. Rearranging for $x_0$:

$$x_0 = \frac{x - u_m t}{1 - \frac{2u_m t}{L}},$$

so $p(x, t) = p(x_0, 0) = \frac{\rho_m}{L - 2u_m t} (x - u_m t)$.

Thus $p(x, t) = \begin{cases} 
0 & \text{if } x < u_m t \\
\frac{\rho_m u_m (x - u_m t)}{L - 2u_m t} & \text{if } u_m t < x < L - u_m t \\
\rho_m & \text{if } x > L - u_m t 
\end{cases}$

After shock ($t > \frac{L}{2u_m}$):

$$p(x, t) = \begin{cases} 
\rho_0 & \text{if } x < L/2 \\
\rho_m & \text{if } x > L/2 
\end{cases}.$$

So:

f) Cars pile up past $x = \frac{L}{2}$, and eventually there are no more cars before $x = \frac{L}{2}$. 
80.1  

a) \[ t = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d}{dx} \left( u_m (1 - \frac{2x}{L}) \right) = -\frac{1}{2u_m} \frac{dx}{dt} \left( 0 \right) = \frac{p_m}{2u_m \left( \frac{\delta_p}{dx} \right)} \]

b) \[ p(x,0) = p_m e^{-x^2/L^2} \]

\[ p(x,t) \]

(1)

The time of first shock occurs when

\[ t = \frac{p_m}{2u_m \left( \frac{-2u_m x}{L^2} e^{-x^2/L^2} \right)} = \frac{-L^2}{4u_m x e^{-x^2/L^2}} \] is minimized

(2) as a function of \( x \), noting that \( x < 0 \) is required for \( t > 0 \). \( t \) is minimized when \( g(x) = x e^{-x^2/L^2} \)

is minimized for \( x < 0 \), \( g' = e^{-x^2/L^2} \left( 1 - \frac{2x^2}{L^2} \right) = 0 \)

so \( g''(x) = e^{-x^2/L^2} \left( \frac{4x^3}{L^2} - \frac{2x}{L^2} + \frac{4x}{L^2} \right) \),

\( x = \frac{L}{\sqrt{2}} \), and there, \( g'' = e^{-x^2/L^2} \left( \frac{4x^2 + 2x}{2} \right) > 0 \), so since \( g \) is concave up & a c.p. of \( g \), and \( g \) approaches \( 0 > \eta \left( \frac{L}{\sqrt{2}} \right) \)

as its argument tends to \( \pm \infty \), the time of the first shock is \( t = \frac{-L^2}{4u_m \left( \frac{\delta_p}{L^2} \right) e^{-L^2}} = \frac{L \delta_p e^{-L^2}}{4u_m} \).

(3) By the work above, this shock first occurs at time \( t = \frac{L \delta_p e^{-L^2}}{4u_m} \), on the characteristic corresponding to \( x_0 = -\frac{L}{\sqrt{2}} \),

which can be found via: \( x' = a = u_m \left( 1 - \frac{2x}{L} \right) \), so \( x(t) = \int u_m \left( 1 - \frac{2x}{L} e^{-\frac{x^2}{L^2}} \right) - \frac{L}{\sqrt{2}}, \) and thus the shock first occurs at location \( x(\frac{L \delta_p e^{-L^2}}{4u_m}) = \frac{L \delta_p e^{-L^2}}{4} (1 - 2 e^{-L^2}) - \frac{L}{\sqrt{2}} \)

\[ = \frac{L}{\sqrt{2}} \left( \frac{L \delta_p}{2} - 2 \right) \).
\[ p(x,0) = \begin{cases} 
  e_1 & x < 0 \\
  e_2 & 0 < x < a \\
  e_3 & a < x 
\end{cases} \quad (e_1 < e_2 < e_3). \]

Two shockwaves at first:

\[ \frac{dx_{s, l}}{dt} = \frac{e_2 u(e_2) - e_1 u(e_1)}{e_2 - e_1} = u_m \left( 1 - \frac{e_1 + e_2}{e_m} \right) \rightarrow \text{greater velocity of the two} \]

\[ \frac{dx_{s, r}}{dt} = \frac{e_3 u(e_3) - e_2 u(e_2)}{e_3 - e_2} = u_m \left( 1 - \frac{e_2 + e_3}{e_m} \right), \text{ so} \]

Intersect @

\[ x^* = \frac{e_2}{a(e_2 - p)} \]

\[ x_{s, l}(t) = t u_m \left( 1 - \frac{e_1 + e_2}{e_m} \right) \]

\[ x_{s, r}(t) = t u_m \left( 1 - \frac{e_2 + e_3}{e_m} \right) + a \]

New \( p \) values to the left & right

of discontinuity in \( p \) @ \((x^*, t^*)\), so

trajectory of shock changes:

\[ \frac{dx_{s, c}}{dt} = \frac{e_3 u(e_3) - e_1 u(e_1)}{e_3 - e_1} = u_m \left( 1 - \frac{e_1 + e_3}{e_m} \right), \text{ so} \]

\[ x_{s, c}(t) = (t - t^*) u_m \left( 1 - \frac{e_1 + e_3}{e_m} \right) + x^* \]

\( t^* = \frac{e_m}{u_m(p_3 - p_1)} \), \( x^* = \frac{a}{e_3 - e_1} (p_m - e_1 - p_2) \)

\( t \leq t^* \) & \( x < x_{s, l}(t) \) or \( t > t^* \) & \( x < x_{s, c}(t) \)

Thus,

\[ p(x,t) = \begin{cases} 
  e_1 & (t \leq t^* \text{ & } x < x_{s, l}(t)) \text{ or } (t > t^* \text{ & } x < x_{s, c}(t)) \\
  e_2 & (t < t^* \text{ & } x_{s, l}(t) < x < x_{s, r}(t)) \text{ or } (t > t^* \text{ & } x_{s, r}(t) < x) \\
  e_3 & (t < t^* \text{ & } x_{s, r}(t) < x) \text{ or } (t > t^* \text{ & } x_{s, c}(t) < x) 
\end{cases} \]

So the two shocks met and produced a new shock.
82.2

\[ p(x, t) \]

\[ \rho \]

\[ \rho(x, t) = \begin{cases} 
\rho_0 & x \leq t u_m (1 - \frac{2e_1}{e_m}) - a \\
\rho_0 e^{\frac{a}{2t}} \left( x - \frac{t u_m (1 - \frac{2e_1}{e_m}) + a}{2} \right) & x \geq \chi_{S,2}(t) 
\end{cases} \]

Shock from \( x_0 = a \): \[ \frac{dx_s}{dt} = u_m \left( 1 - \frac{\rho_0 + p_1}{\rho_m} \right) \], so \[ x_{S,1}(t) = t u_m \left( 1 - \frac{\rho_0 + p_1}{\rho_m} \right) + a \].

New density to the left of shock when characteristic from \( x_0 = a \) hits the shock: \[ t u_m \left( 1 - \frac{2\rho}{\rho_m} \right) - a = t u_m \left( 1 - \frac{\rho_0 + p_1}{\rho_m} \right) + a \], so this happens at time \[ t^* = \frac{2a \rho_m}{u_m (\rho - \rho_0)} \], at location \[ x^* = x_{S,1}(t^*) = \left( \frac{2a \rho_m}{u_m (\rho - \rho_0)} \right) u_m \left( 1 - \frac{\rho_0 + p_1}{\rho_m} \right) + a = \left( \frac{2a}{\rho_1 - \rho_0} \right) (\rho_m - \rho_0 - p_1) \].

Now, the fan characteristics are given by \( x = t u_m \left( 1 - \frac{2\rho}{\rho_m} \right) - a \), or: \( \rho = \frac{\rho_m}{2} \left( 1 - \frac{x + a}{t u_m} \right) \). So just to the left of the new (nonlinear) shockwave, we use this density:

\[ \frac{dx_{S,2}}{dt} = \frac{p_m u(p_m) - \rho_u(u_1)}{\rho_1 - \rho_1} = u_m \left( 1 - \frac{\rho_0 + p_1}{\rho_m} \right) - u_m \frac{p_0}{\rho_m} = u_m \frac{\rho_0 - p_0}{\rho_m} \]

\[ = u_m \left( 1 - \frac{\rho_0}{\rho_m} \right) - u_m \frac{\rho_0}{\rho_m} \left( 1 - \frac{x_{S,2} + a}{t u_m} \right) \]

Thus, \[ x_{S,2}(t) = \left[ u_m \left( \frac{1}{2} - \frac{\rho_0}{\rho_m} \right) + \frac{a}{2t} \right] e^{\frac{a}{2t}} dt \]

Factors to solve lead, obtaining \[ x_{S,2}(t) = u_m \left( 1 - \frac{2\rho_0}{\rho_m} \right) + a t - \frac{a}{2} t^{1/2} + A \]

where \( A \) must be chosen s.t. the initial condition \( x_{S,2}(t^*) = x^* \) is satisfied.
The first portion of shock:

\[
\frac{dx}{dt} = \frac{e_i u(e_i) - p_0 u(p_0)}{e_i - p_0} = u_m \left(1 - \frac{e_0 + e_i}{p_m}\right) ,
\]

so

\[
x_{s_1}(t) = u_m \left(1 - \frac{e_0 + e_i}{p_m}\right) - a .
\]

In rarefied region, density again given by

\[
x = \frac{u_m}{u_m(1 - \frac{x-a}{p_m})},
\]

which intersects \(x_{s_1}\), when

\[
t_* = \frac{2a e_m}{u_m(b_0 - e_0)} ,
\]

at location

\[
x_* = x_{s_1}(t_*) = \frac{2a e_m}{(p_0 - e_0)} \left(1 - \frac{e_0}{p_m}\right) - a .
\]

After that time, the solution

\[
x_{s_2}(t) = \frac{u_m \left(1 - \frac{e_i}{p_m}\right) + \frac{x_{s_2} - a}{2t}}{\frac{e_i - e_0}{2t}} = 2u_m \left(\frac{1}{2} - \frac{e_i}{p_m}\right) t^{1/2} + at^{1/2} + \frac{e_i}{2u_m} t - \frac{a e_i}{u_m}
\]

\[
= 2a u_m \left(\frac{1}{2} - \frac{e_i}{p_m}\right) t + a + At^{1/2},
\]

where \(A\) must be chosen so that \(x_{s_2}(t^*) = x^*\). Then,

\[
(t < t^* & x < x_{s_1}(t)) \text{ or } (t \geq t^* & x < x_{s_2}(t))
\]

\[
\rho(x,t) = \begin{cases} 
  e_i & (t < t^* & x < x_{s_1}(t)) \text{ or } (t \geq t^* & x < x_{s_2}(t)) \\
  e_0 & (t < t^* & x_{s_1}(t) < x < u_m(1 - \frac{e_0}{p_m}) + a) \text{ or } (t \geq t^* & x < u_m(1 - \frac{e_0}{p_m}) + a) \\
  e_m \left(1 - \frac{x-a}{u_m}\right) & (t < t^* & u_m(1 - \frac{e_0}{p_m}) + a \leq x \leq u_m(1 - \frac{2a}{p_m}) + a) \text{ or } (t \geq t^* & x < u_m(1 - \frac{2a}{p_m}) + a) \\
  e_1 & u_m(1 - \frac{2a}{p_m}) + a < x
\end{cases}
\]
84.1 Looking on pg. 351 of the text for inspiration, we modify the conservation of cars equation in the center of the page to look like:

\[ \frac{d}{dt} \int_{x_1}^{x_2} p \, dx = q(x_1, t) - q(x_2, t) + \beta_0 (x_2 - x_1) \]

where \( \frac{d}{dt} \) is the net rate of entering cars per mile.

\[ \Rightarrow [p] \frac{dx_5}{dt} + \int_{x_1}^{x_2} p_t \, dx + \int_{x_1}^{x_2} p_x \, dx = q(x_1, t) - q(x_2, t) + \beta_0 (x_2 - x_1) \]

which is a modified equation (77.5) (the derivation to this point is nearly identical). Then taking \( x_1 \to x_3 \), \( x_2 \to x_5 \) and noting that \([p] \) does not shrink to 0 as a result of this, the \( \beta_0 (x_2 - x_1) \) term in the above equation vanishes leaving:

\[ [e] \frac{dx_5}{dt} = [q] \]

and thus \( \frac{dx_5}{dt} = \frac{[q]}{[p]} \), the same condition describing the shock velocity as before.

84.2 Since the new characteristics, \( x = u_m (1 - \frac{2 \beta_0}{\rho_m} t) - \frac{u_m}{\rho_m} t^2 + x_0 \), are identical to the old characteristics (with \( \beta_0 = 0 \)), \( x = u_m (1 - \frac{2 \beta_0}{\rho_m} t) + x_0 \), except for the additional \( (\frac{2 \beta_0}{\rho_m} t^2) \) term (which is independent of \( \beta_0 \)), we get that any two characteristics, \( x_1(t) = u_m (1 - \frac{2 \beta_1}{\rho_m} t) - \frac{u_m}{\rho_m} t^2 + x_1 \) and \( x_2(t) = u_m (1 - \frac{2 \beta_2}{\rho_m} t) - \frac{u_m}{\rho_m} t^2 + x_2 \), will intersect (if at all...)

\[ u_m (1 - \frac{2 \beta_1}{\rho_m} t) - \frac{u_m}{\rho_m} t^2 + x_1 = u_m (1 - \frac{2 \beta_2}{\rho_m} t) - \frac{u_m}{\rho_m} t^2 + x_2 \]

which is the same condition for the corresponding characteristics in the case \( \beta_0 = 0 \) to intersect! So the time of neighboring characteristics to first intersect must also be the same. Also \( \beta \) can be picked so that \( \beta < 0 \) as the \( x \)-location of the intersection
85.1 Clearly, none of the characteristics in the regions of the characteristic diagram where characteristics are straight lines intersect. This is because these lines are either parallel, or their slopes increase with t. Thus, we only need to check that characteristics in the region where \( f = f_0 \) do not intersect. Since the characteristics stemming from \( x_0 = 0, t_0 = T \) are vertical translates of each other and are one-to-one, they cannot intersect. Also, the characteristics stemming from \( 0 \leq x_0 \leq x_E, t_0 = 0 \) are horizontal translates of each other and are well-defined functions of x (i.e., only one x-value for each t-value), so these characteristics also cannot intersect. We could also check that characteristics from one region do not intersect characteristics from the other region, but then these characteristics wouldn't be neighbors, so we are done.

85.2 a) Now, by the same derivation as in eqn. 85.1 of the text, the parabolic characteristic will double-back on themselves. Since equation characteristics will double-back on themselves, since equation 85.5 is still valid (for \( p < \frac{p_m}{2} \), since that's where \( 0 = q' = x \)) and equations 85.2b are still valid (up until the time of the first shock) and they have the same slope at \( x = x_E, t = t_E \). This helps us draw our new characteristic diagram:

\[
p = \frac{v_E}{2} + \frac{p_m}{4} - \frac{1}{4} \left( \frac{2v_E + p}{p_m} \right)^2 - \frac{2p_m}{v_m} \left( \frac{u_m + x_E}{x_E} \right).
\]

This ends up being!
b) A shock first occurs when the parabolic characteristic emanating from $x_0 = 0, t_0 = 0$ starts to double-back on itself, since this is when the characteristic is no longer one-to-one, and neighboring characteristics just above it will intersect with it at that time. Since that parabola is given by $x = u_m t - \rho_0 \frac{u_m}{p_m} t^2$, we have $0 = x' = u_m - 2\rho_0 \frac{u_m}{p_m} t$, implying $t^* = \frac{p_m}{2\rho_0}$ is the time of the first shock. The $x$-coordinate of this first shock is: $x^* = \frac{u_m p_m}{4\rho_0} < x_E$

c) As derived in problem 84.1, the shock obeys the same ODE in the case of constantly entering/exiting cars as in the case of no entering/exiting:

$$\frac{dx}{dt} = \frac{\rho(x_s, t) u(x_s, t) - \rho(x_s, t) u(x_s, t)}{\rho(x_s, t) - \rho(x_s, t)},$$

where $\rho(x_s, t) = t \rho_0$ (since the characteristics to the right of $(x_s(t), t)$ stem from $0 \leq x \leq x_E, t > 0$) and $\rho(x_s, t) = (t - T) \rho_0$, where $T$ is obtained by isolating $T$ in equation 85.3b (using $x_s$ in place of $x$), selecting the larger of the two roots.

The initial condition for the shock is $x_s(t^*) = x^*$, where $x^*$ and $t^*$ are given in part (b).

d) For $t u_m - t^2 \frac{u_m}{p_m} \leq x \leq x_E, 0 \leq t \leq t^* = \frac{p_m}{2\rho_0}$, we have $p = \rho_0$, as those corresponding characteristics all emanate from $0 \leq x \leq x_E, t = 0$. For $0 \leq x \leq t u_m - t^2 \frac{u_m}{p_m}$, $0 \leq t \leq t^* = \frac{p_m}{2\rho_0}$, we have $p = \rho_0(t - T)$, where again $T$ is obtained (as a function of $x, t$) by isolating $T$ from equation 85.3b, selecting the larger of the two roots.

Combining these with our answer to part (a) & noting that $\rho = 0$ for $x < 0$, we have specified $p(x, t)$ for all times $t \leq t^*$, before the first shock.