# Large Girth and Chromatic Number

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## 1 Introduction

Our main result will be to show that there exist graphs with both arbitrarily large girth and chromatic number. While we will not explicitly construct such graphs, we will show that they exist using the *probabilistic method*, a powerful tool in combinatorics. We will first define a few (possibly new) graph theory terms and prove some simple results involving them. We then (non-rigorously) define a suitable (probability) space of *random graphs* with which we will prove a few important lemmas involving expected properties of random graphs. Finally, we will bring everything together and show that graphs with large girth and chromatic number exist with positive probability, giving us our desired result.

## 2 Graph Theory Definitions

We will only be considering connected, simple graphs throughout. We will assume that a graph $G = (V, E)$ is a defined by a vertex set $V$ and an edge set $E \subseteq [V]^2$ (That is, every element of $E$ is a two-element subset of $V$). Also, if $x, y \in V$ and $\{x, y\} \in E$, we will say that the edge $xy$ is in the graph $G$.

**Definition 2.1.** *The **girth** of a graph $G$ is the length of the shortest cycle. It is denoted by $g(G)$.*

**Definition 2.2.** *The **chromatic number** of a graph $G$ with vertex set $V$ and edge set $E$ is the smallest $n \in \mathbb{N}$ such that there exists a function $f : V \to \{0, 1, \ldots, n-1\}$ such that if $\{x, y\} \in E$, then $f(x) \neq f(y)$. This number if denoted by $\chi(G)$.*

**Definition 2.3.** *The **independence number** of a graph $G$ with vertex set $V$ is the maximum cardinality of a subset $U \subseteq V$ such that for all $x, y \in U$, the edge $\{x, y\} \notin E$. This is denoted by $\alpha(G)$.*

**Example 2.4.** *Let $C_n$ denote the cycle with $n$ vertices. If $n$ is even, then $\chi(C_n) = 2$, but if $n$ is odd, then $\chi(C_n) = 3$. Let $K_n$ denote the complete graph on $n$ vertices. Then, $\chi(K_n) = n$.*
*It is easy to verify that $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha(K_n) = 1$.*

**Lemma 2.5.** *If $G$ is a graph with $n$ vertices, then $n \leq \alpha(G)\chi(G)$.*

*Proof.* Color $G$ with $\chi(G)$ colors. Since the set of vertices of any given color is independent, they each have size at most $\alpha(G)$. Thus, $n \leq \alpha(G)\chi(G)$. □

## 3 Random Graphs

Let $V$ be a fixed set of elements, say, $\{0, 1, 2, \ldots, n-1\}$. In order to use probability, we need to turn the set $\mathcal{G}$ of all graphs on $V$ into a probability space. We will use $\mathcal{G}(n, p)$ to denote the space of random graphs with $n$ vertices, where $p$ denoted the probability that each edge is in an event (graph) of $\mathcal{G}(n, p)$. It is important to note that the edge events are independent of one another. Also, define $q := 1 - p$. We will better illustrate this concept while laying the groundwork for the main theorem.

**Lemma 3.1.** *For all integers $n, k$ with $n \geq k \geq 2$, the probability that $G \in \mathcal{G}(n, p)$ has a set of $k$ independent vertices is at most*

$$P[\alpha(G) > k] \leq \binom{n}{k} q^{\binom{k}{2}}.$$

*Proof.* Consider a fixed $k$-set of vertices. The probability that it is an independent set is $q^{\binom{k}{2}}$. Note that there are exactly $\binom{n}{k}$ such $k$-sets. Thus, the probability that at least one $k$-subset of $n$ vertices is independent is at most $\binom{n}{k}q^{\binom{k}{2}}$. (This follows from the inclusion-exlusion principle) $\square$

**Lemma 3.2** (Markov's Inequality)**.** *Let $X \geq 0$ be a random variable on $\mathcal{G}(n,p)$ and $a > 0$. Then*

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

*Proof.*

$$\begin{aligned} E[X] &= \sum_{G \in \mathcal{G}(n,p)} P[\{G\}] \cdot X(G) \\ &\geq \sum_{\substack{G \in \mathcal{G}(n,p) \\ X(G) \geq a}} P[\{G\}] \cdot X(G) \\ &\geq \sum_{\substack{G \in \mathcal{G}(n,p) \\ X(G) \geq a}} P[\{G\}] \cdot a \\ &= P[X \geq a] \cdot a. \end{aligned}$$

Thus, $P[X \geq a] \leq \frac{E[X]}{a}$. $\square$

**Definition 3.3.** *The number is sequences of $k$ distinct elements of a given $n$-set is*

$$(n)_k := n(n-1)(n-2)\cdots(n-k+1).$$

**Lemma 3.4.** *The expected number of $k$-cycles (cycles of length $k$) in $G \in \mathcal{G}(n,p)$ is*

$$E[X] = \frac{(n)_k}{2k}p^k.$$

*Proof.* For every $k$-cycle $C$ we define the indicator random variable of $C$, namely, $X_C : \mathcal{G}(n,p) \to \{0,1\}$ where

$$X_C : G \mapsto \begin{cases} 1 & \text{if } C \subseteq G \\ 0 & \text{otherwise.} \end{cases}$$

Note that $E[X_C] = P[C \subseteq G] = p^k$. We must now count how many $k$-cycles there are. There are $(n)_k$ sequences of distinct vertices in $V$, and each cycle can be identified by $2k$ of those cycles (because we can represent cycles as a list of vertices, choosing any of the $k$ vertices to start and then one of two directions to list), so there are exactly $\frac{(n)_k}{2k}$ cycles. Consider a random variable $X$ which assigns each graph $G \in \mathcal{G}(n,p)$ its number of $k$-cycles. Then, we have

$$X = \sum_C X_C.$$

By linearity of expectation, we see that

$$E[X] = E[\sum_C X_C] = \sum_C E[X_C] = \frac{(n)_k}{2k}p^k,$$

the desired result. $\square$

**Lemma 3.5.** *Let $k > 0$ be an integer, and let $p = p(n)$ by a function of $n$ such that $p \geq \frac{6k \ln n}{n}$ for $n$ large. Then*

$$\lim_{n\to\infty} P[\alpha \geq \frac{n}{2k}] = 0.$$

*Proof.* For all integers $n, r$ with $n \geq r \geq 2$, Lemma 3.1 implies that

$$\begin{aligned} P[\alpha \geq r] &\leq \binom{n}{r} q^{\binom{r}{2}} \\ &\leq n^r q^{\binom{r}{2}} \\ &= (nq^{\frac{r-1}{2}})^r \\ &\leq (ne^{\frac{-p(r-1)}{2}})^r. \end{aligned}$$

Note that the last inequality comes from the fact that $1 - p \leq e^{-p}$ for all $p$. Now suppose that $p \geq \frac{6k \ln n}{n}$ and $r \geq \frac{n}{2k}$. Then,

$$\begin{aligned} ne^{\frac{-p(r-1)}{2}} &= ne^{\frac{-pr}{2} + \frac{p}{2}} \\ &\leq ne^{\frac{-3}{2} \ln n + \frac{p}{2}} \\ &\leq nn^{\frac{-3}{2}} e^{\frac{1}{2}} \\ &= \frac{\sqrt{e}}{\sqrt{n}}. \end{aligned}$$

Note that $\lim_{n \to \infty} = \frac{\sqrt{e}}{\sqrt{n}}$. Since $p \geq \frac{6k \ln n}{n}$ for $n$ large, by letting $r := \lceil \frac{n}{2k} \rceil$ we see that

$$\lim_{n \to \infty} P[\alpha \geq \frac{n}{2k}] = \lim_{n \to \infty} P[\alpha \geq r] = 0.$$

□

## 4 The Probabilistic Method

Using the results of the previous section, it will be stragithforward to prove the following theorem.

**Theorem 4.1.** *For every integer $k$ there exists a graph $H$ with girth $g(H) > k$ and chromatic number $\chi(H) > k$.*

*Proof.* Assume that $k \geq 3$. Fix $\epsilon > 0$ such that $0 < \epsilon < \frac{1}{k}$, and set $p := n^{\epsilon - 1}$. Define $X(G)$ to be the number of short cycles (cycles of length at most $k$) in a random grapn $G \in \mathcal{G}(n, p)$. By Lemma 3.4, we have

$$E[X] = \sum_{i=3}^{k} \frac{(n)_i}{2i} p^i \leq \frac{1}{2} \sum_{i=3}^{k} n^i p^i \leq \frac{1}{2}(k-2) n^k p^k.$$

Note that $(np)^i \leq (np)^k$ because $np = n \cdot n^{\epsilon - 1} = n^\epsilon \geq 1$. Thus, by Markov's Inequality (Lemma 3.2), we have

$$\begin{aligned} P[X \geq \frac{n}{2}] &\leq \frac{E[X]}{\frac{n}{2}} \\ &\leq \frac{\frac{1}{2}(k-2) n^k p^k}{\frac{n}{2}} \\ &= (k-2) n^{k-1} p^k \\ &= (k-2) n^{k-1} n^{(\epsilon - 1)k} \\ &= (k-2) n^{k\epsilon - 1}. \end{aligned}$$

Recall that by our choice of $\epsilon$, we have $k\epsilon - 1 < 0$. Thus,

$$\lim_{n \to \infty} P[X \geq \frac{n}{2}] = 0.$$

It follows from Lemma 3.5 that we can choose $n_1$ large enough that $P[\alpha \geq \frac{n}{2k}] < \frac{1}{2}$. By the above, we can also choose $n_2$ large enough such that $P[X \geq \frac{n}{2}] < \frac{1}{2}$. Choose $n \geq \max\{n_1, n_2\}$. We now know that there exists a graph $G \in \mathcal{G}(n, p)$ with fewer than $\frac{n}{2}$ short cycles and $\alpha(G) < \frac{n}{2k}$. From each short cycle in $G$, delete one vertex, and call this new graph $H$. Note that $|H| \geq \frac{n}{2}$. Also, $g(H) > k$. Finally, by Lemma 2.5,

$$\chi(H) \geq \frac{|H|}{\alpha(H)} \geq \frac{\frac{n}{2}}{\alpha(G)} > \frac{\frac{n}{2}}{\frac{n}{2k}} = k.$$

□