MATH 3B: HOMEWORK 4, SECTION 7.3, #49

Now that I have remembered how to multiply and do partial fractions, lets look at problem 49. We want to compute $\int \frac{1}{(x^2-9)^2}\,dx$, so let us use the partial fraction decomposition. Since $(x^2-9)^2 = (x+3)^2(x-3)^2$, we may write

$$\frac{1}{(x+3)^2(x-3)^2} = \frac{Ax+B}{(x+3)^2} + \frac{Cx+D}{(x-3)^2}.$$

Rewriting the right hand side over a single denominator gives

$$\frac{1}{(x+3)^2(x-3)^2} = \frac{(Ax+B)(x-3)^2 + (Cx+D)(x+3)^2}{(x+3)^2(x-3)^2}.$$

We expand and equate the numerators to get

$$1 = Ax^3 + (B-6A)x^2 + (9A-6B)x + 9B + Cx^3(6C+D)x^2(9C+6D)x + 9D.$$

We equate the coefficients of the different powers of x on the left and right side of this equation to get

$$A + C = 0 \tag{1}$$
$$(B-6A) + (6C+D) = 0 \tag{2}$$
$$(9A-6B) + (9C+6D) = 0 \tag{3}$$
$$9B + 9D = 1. \tag{4}$$

The first and last equation look the simplest, so let use rewrite them as $C = -A$ and $D = (1-9B)/9$. We then plug these into equations (2) and (3) to get

$$B - 6A + 6(-A) + \frac{1-9B}{9} = 0 \tag{5}$$
$$9A - 6B + 9(-A) + 6\left(\frac{1-9B}{9}\right) = 0. \tag{6}$$

Once we collect like terms in equations (5) and (6) (and multiply by 9 to get remove the fraction) we get

$$1 - 108A = 0 \tag{7}$$
$$18B - 1 = 0. \tag{8}$$

Conveniently, these equations pretty much solve themselves, since A disappears in (8) and B disappears in (7). Looking back at equations (1) and (4), combined with (7) and (8) we see that

$$A = \frac{1}{108} \qquad B = \frac{1}{18}$$
$$C = \frac{-1}{108} \qquad D = \frac{1}{18}.$$

Now that we have these values, we see that

$$\frac{1}{(x+3)^2(x-3)^2} = \frac{\frac{x}{108}+\frac{1}{18}}{(x+3)^2} + \frac{\frac{-x}{108}+\frac{1}{18}}{(x-3)^2}.$$

To make this simpler, we factor out the 1/108. Remember, we are trying to solve an integral, so we want to compute:

$$\int \frac{1}{(x^2-9)^2}\,dx = \frac{1}{108}\left(\int \frac{x+6}{(x+3)^2} + \frac{-x+6}{(x-3)^2}\,dx\right).$$

Let us look at the two integrals separately. We did examples like this in class, so I will go quickly (the integrals are good practice, you should try them on your own):

$$\int \frac{x+6}{(x+3)^2}\,dx = \ln|x+3| - \frac{3}{x+3}$$

and

$$\int \frac{-x+6}{(x-3)^2}\,dx = -\ln|x-3| - \frac{3}{x-3}.$$

Plugging these in above gives us

$$\begin{aligned}\int \frac{1}{(x^2-9)^2}\,dx &= \frac{1}{108}\left[\left(\ln|x+3| - \frac{3}{x+3}\right) + \left(-\ln|x-3| - \frac{3}{x-3}\right)\right]\\ &= \frac{1}{108}\left(\ln\left|\frac{x+3}{x-3}\right| - \frac{3}{x+3} - \frac{3}{x-3}\right).\end{aligned}$$

In class I broke up the partial fraction a little differently, that doest give the same answer, but the system of equations is a little longer to solve (there are two extra steps).

Let us see quickly why they both give the same answer. I wrote in class

$$\frac{1}{(x+3)^2(x-3)^2} = \frac{A'}{x+3} + \frac{B'}{(x+3)^2} + \frac{C'}{x-3} + \frac{D'}{(x-3)^2}.$$

Why did I put A' instead of A? These values will be different! The variables represent different quantities as earlier. But, we can go backwards a little bit and see that

$$\frac{A'}{x+3} + \frac{B'}{(x+3)^2} = \frac{A'x + (3A'+B')}{(x+3)^2} \quad \text{and} \quad \frac{C'}{x-3} + \frac{D'}{(x-3)^2} = \frac{C'x + (-3C'+D')}{(x-3)^2}.$$

We see from this that it is essentially solving the same problem, we just get different values for B, D and B', D'.

I recommend trying a bunch of different problems, with different types of polynomials in the denominator. This is the best way to get a feel for how to do these problems. If you need to check your answers, try "The Integrator" (www.integrals.wolfram.com).