Patching and Multiplicity $2^k$ for Shimura Curves

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Abstract

We use the Taylor–Wiles–Kisin patching method to investigate the multiplicities with which Galois representations occur in the mod $\ell$ cohomology of Shimura curves over totally real number fields. Our method relies on explicit computations of local deformation rings done by Shotton, which we use to compute the Weil class group of various deformation rings. Exploiting the natural self-duality of the cohomology groups, we use these class group computations to precisely determine the structure of a patched module in many new cases in which the patched module is not free (and so multiplicity one fails).

Our main result is a "multiplicity $2^k$" theorem in the minimal level case (which we prove under some mild technical hypotheses), where $k$ is a number that depends only on local Galois theoretic information at the primes dividing the discriminant of the Shimura curve. Our result generalizes Ribet’s classical multiplicity 2 result and the results of Cheng, and provides progress towards the Buzzard-Diamond-Jarvis local-global compatibility conjecture. We also prove a statement about the endomorphism rings of certain modules over the Hecke algebra, which may have applications to the integral Eichler basis problem.

1 Introduction

One of the most powerful tools in the study of the Langlands program is the Taylor–Wiles–Kisin patching method which, famously, was originally introduced by Taylor and Wiles [Wil95, TW95] to prove Fermat’s Last Theorem, via proving a special case of Langlands reciprocity for GL$_2$.

In its modern formulation (due to Kisin [Kis09b] and others) this method considers a ring $R_\infty$, which can be determined explicitly from local Galois theoretic data, and constructs a maximal Cohen–Macaulay module $M_\infty$ over $R_\infty$ by gluing together various cohomology groups. Due to its construction, $M_\infty$ is closely related to certain automorphic representations, and so determining its structure has many applications in the Langlands program beyond simply proving reciprocity.

A few years after Wiles’ proof, Diamond [Dia97] discovered that patching can also be used to prove mod $\ell$ multiplicity one statements in cases where the $q$-expansion principle does not apply. In his argument he considers a case when the ring $R_\infty$ is formally smooth, and so the Auslander-Buchsbaum formula allows him to show that $M_\infty$ is free over $R_\infty$, a fact which easily implies
multiplicity one. There are however, many situations arising in practice in which $R_\infty$ is not formally smooth, and so Diamond’s method cannot be used to determine multiplicity one statements.

In this paper, we introduce a new method for determining the structure of a patched module $M_\infty$ arising from the middle degree cohomology of certain Shimura varieties, which applies in cases when $R_\infty$ is Cohen–Macaulay, but not necessarily formally smooth. Using this, we are able to compute the multiplicities for Shimura curves over totally real number fields in the minimal level case, under some technical hypotheses. Our main result is the following (which we state here, using some notation and terminology which we define later):

**Theorem 1.1.** Let $F$ be a totally real number field, and let $D/F$ be a quaternion algebra which is ramified at all but at most one infinite place of $F$. Take some irreducible Galois representation $\overline{\rho}: G_F \to \text{GL}_2(\mathbb{F}_\ell)$, where $\ell > 2$ is a prime which is unramified in $F$, and prime to the discriminant of $D$. Assume that:

1. $\overline{\rho}$ is automorphic for $D$.
2. $\overline{\rho}|_{G_{F(v)}}$ is finite flat for all primes $v|\ell$ of $F$.
3. If $v$ is any prime of $F$ at which $D$ ramifies, then $\overline{\rho}$ is Steinberg at $v$ and $\text{Nm}(v) \not\equiv -1 \pmod{\ell}$.
4. The restriction $\overline{\rho}|_{G_{F(\zeta_\ell)}}$ is absolutely irreducible.

Let $K_{\text{min}} \subseteq D^\times(K_0)$ be the minimal level at which $\overline{\rho}$ occurs, and let $X^D(K_{\text{min}})$ be the Shimura variety (of dimension either 0 or 1) associated to $K_{\text{min}}$. Then the multiplicity $^1$ with which $\overline{\rho}$ occurs in the mod $\ell$ cohomology of $X^D(K_{\text{min}})$ is $2^k$, where

$$k = \# \left\{ v \mid D \text{ ramifies at } v, \overline{\rho} \text{ is unramified at } v \text{ and, } \overline{\rho}(\text{Frob}_v) \text{ is a scalar} \right\}.$$

**Remark.** This was first proven by Ribet [Rib90] in the case when $F = \mathbb{Q}$, $D = D_{pq}$ is the indefinite quaternion ramified at two primes, $p$ and $q$, and $K_{\text{min}} = D_{pq}(\hat{\mathbb{Z}})$, i.e. the “level one” case (in which case $k$ is forced to be either 0 or 1). He also proved a more general result in the case when $F = \mathbb{Q}$ and $D = D_p$ is the definite quaternion algebra ramified at one prime, $p$.

Yang [Yan96] gave a (non-sharp) upper bound on the multiplicity in the case where $F = \mathbb{Q}$ and $\overline{\rho}$ was ramified in at least half of the primes in the discriminant of $D$, and also showed that multiplicities of at least 4 are achievable. Helm [Hel07] strengthened this result to prove the optimal upper bound of $2^k$ on the multiplicity, again in the case of $F = \mathbb{Q}$, but without the ramification condition for $\overline{\rho}$.

Cheng [Che] extended some of these results to the case when $F$ is arbitrary, and showed that a multiplicity of $2^k$ is possible for any $k$.

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$^1$In the case when $D$ unramified at exactly one infinite place of $F$, $X^D(K_{\text{min}})$ is an algebraic curve, and so this multiplicity is just the number of copies of $\overline{\rho}$ which appear in the étale cohomology group $H^1_{\text{et}}(X^D(K_{\text{min}}),\mu_\ell)$. In the case when $D$ is ramified as all infinite pales of $F$, $X^D(K_{\text{min}})$ is just a discrete set of points and so $\overline{\rho}$ does not actually appear in the cohomology. In this case, by the multiplicity we just mean the dimension of the eigenspace $H^0(X^D(K_{\text{min}}),F_\ell)[m]$ for $m$ the corresponding maximal ideal of the Hecke algebra.
Based on the results over $\mathbb{Q}$, Buzzard, Diamond and Jarvis [BDJ10] gave a conjectural mod $\ell$ local-global compatibility conjecture, which gives a conjectural description of the multiplicity for arbitrary $F$, $D$ and (prime to $\ell$) level. Theorem 1.1 is a special case of this conjecture.

The previous results relied heavily on facts about integral models of Shimura curves, as well as other results such as mod $\ell$ multiplicity one statements for modular curves (arising from the $q$-expansion principle) and Ihara’s Lemma. Our approach is entirely different, and does not rely on any such statements about Shimura curves.

Our method relies on the natural self-duality of the module $M_\infty$, combined with an explicit calculation of the ring $R_\infty$ arising in the patching method, together with its Weil class group. While these computations may be quite difficult in higher dimensions, all of the relevant local deformation rings have been computed by Shotton [Sho16] in the $GL_2$ case, and moreover his computations show that the ring $R_\infty/\lambda$ is (the completion of) the ring of functions on a toric variety. This observation makes it fairly straightforward to apply our method in the $GL_2$ case, and hence to precisely determine the structure of the patched module $M_\infty$.

Additionally, our explicit description of the patched module $M_\infty$ allows us to extract more refined data about the Hecke module structure of the cohomology groups, beyond just the multiplicity statements (see Theorem 1.2, below). This has potential applications to the integral Eichler basis problem.

Many of the conditions in the statement of Theorem 1.1 were included primarily to simplify the proof and exposition, and are not fundamental limitations on our method.

Condition (2) is essentially an assumption that the minimal level of $\mathfrak{p}$ is prime to $\ell$. It, together with the earlier assumption that $\ell$ does not ramify in $F$, is included to ensure that the local deformation rings $R^{\square, fl, \psi}(\mathfrak{p}|_{G_v})$ considered in Section 2 are formally smooth. As the local deformation rings at $v|\ell$ are known to be formally smooth in more general situations, this condition can likely be relaxed somewhat with only minimal modifications to our method. Even more generally, it is likely that our techniques can be extended to certain other situations in which the local deformation rings at $v|\ell$ are not formally smooth, provided we can still explicitly compute these rings.

Condition (3) ensures that the Steinberg deformation ring, $R^{\square, st, \psi}(\mathfrak{p}|_{G_v})$ from Section 2 is a domain. As the ring $R^{st, \square, \psi}(\mathfrak{p}|_{G_v})$ has been computed by Shotton [Sho16] in all cases, it is likely that a more careful analysis of the excluded case will allow us to remove this condition as well.

The restriction to the minimal level is similarly intended to ensure that the deformation rings considered will be domains. It is possible this restriction can be relaxed in certain cases, particularly in cases when Ihara’s Lemma is known.

We intend to explore the possibility of relaxing or removing some of these conditions in future work.

Lastly, condition (4) is the classical “Taylor–Wiles condition”\(^2\), which is a technical condition

\(^2\)Experts will note that there is also another Taylor–Wiles condition one must assume in the case when $\ell = 5$ and $\sqrt{5} \in F$. In our case however, this situation is already ruled out by the assumption that $\ell$ is unramified in $F$, and so...
necessary for our construction in Section 4. It is unlikely that this condition can be removed without a significant breakthrough.

1.1 Definitions and Notation

Let $F$ be a totally real number field, with ring of integers $\mathcal{O}_F$. We will always use $v$ to denote a finite place $v \subseteq \mathcal{O}_F$. For any such $v$, let $F_v$ be the completion of $F$ and let $\mathcal{O}_{F,v}$ be its ring of integers. Let $\varpi_v$ be a uniformizer in $\mathcal{O}_{F,v}$ and let $k_v = \mathcal{O}_{F,v}/\varpi_v = \mathcal{O}_F/v$ be the residue field. Let $\text{Nm}(v) = \#k_v$ be the norm of $v$.

Let $D$ be a quaternion algebra over $F$ with discriminant $\mathfrak{D}$ (i.e. $\mathfrak{D}$ is the product of all finite primes of $F$ at which $D$ is ramified). Assume that $D$ is either ramified at all infinite places of $F$ (the totally definite case), or split at exactly one infinite place (the indefinite case).

Now fix a prime $\ell > 2$ which is relatively prime to $D$ and does not ramify in $F$. For the rest of this paper we will fix a finite extension $E/\mathbb{Q}_\ell$. Let $\mathcal{O}$ be the ring of integers of $E$, $\lambda \in \mathcal{O}$ be a uniformizer and $F = \mathcal{O}/\lambda$ be its residue field.

For any $\lambda$-torsion free $\mathcal{O}$-module $M$, we will write $M^\vee = \text{Hom}_{\mathcal{O}}(M, \mathcal{O})$ for it’s dual.

We define a level to be a compact open subgroup $K = \prod_{v \subseteq \mathcal{O}_F} K_v \subseteq \prod_{v \subseteq \mathcal{O}_F} D^\times(\mathcal{O}_{F,v}) \subseteq D^\times(\mathbb{A}_F)$ where we have $K_v = D^\times(\mathcal{O}_{F,v})$ for each $v \mid \mathfrak{D}$. We say that $K$ is unramified at some $v \not\mid \mathfrak{D}$ if $K_v = \text{GL}_2(\mathcal{O}_{F,v})$. Note that $K$ is necessarily unramified at all but finitely many $v$. Write $N_K$ for the product of all places $v \mid \mathfrak{D}$ where $K$ is ramified.

If $D$ is totally definite, let

$$S^D(K) = \{f : D^\times(F) \backslash D^\times(\mathbb{A}_F) / K \to \mathcal{O}\}.$$ 

If $D$ is indefinite, let $X^D(K)$ be the Riemann surface $D^\times(F) \backslash (D^\times(\mathbb{A}_F) \times \mathcal{H}) / K$ (where $\mathcal{H}$ is the complex upper half plane). Give $X^D(K)$ its canonical structure as an algebraic curve over $F$, and let $S^D(K) = H^1(X^D(K), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}$. Also define $\overline{S^D}(K) = S^D(K) \otimes_{\mathcal{O}} \mathbb{F}$.

For any finite prime ideal $v$ of $F$ for which $v \nmid \mathfrak{D}N_K$, consider the double-coset operators $T_v, S_v : S^D(K) \to S^D(K)$ given by

$$T_v = \left[ K \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} K \right], \quad S_v = \left[ K \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} K \right]$$

Let

$$T^D(K) := \mathcal{O} \left[ T_v, S_v \mid v \subseteq \mathcal{O}_F, v \nmid \mathfrak{D}N_K \right] \subseteq \text{End}_\mathcal{O}(S^D(K))$$

we do not need to explicitly rule it out.
be the (anemic) Hecke algebra.

It will sometimes be useful to treat the $T^D(K)$’s as quotients of a fixed ring $\mathbb{T}_{S,\text{univ}}^\text{univ} := \mathcal{O}[T_v, S_v^{\pm 1}]_{v \not\in S}$, where $S$ is a finite set of primes, containing all primes dividing $\mathfrak{D} N_K$ (here, $T_v$ and $S_v$ are treated as commuting indeterminants). We can thus think of any maximal ideal $m \subseteq T^D(K)$ as being a maximal ideal of $\mathbb{T}_{S,\text{univ}}^\text{univ}$, and hence as being a maximal ideal of $T^D(K')$ for all $K' \subseteq K$.

Now let $G_F = \text{Gal}(\overline{\mathbb{Q}}/F)$ be the absolute Galois group of $F$. For any $v$, let $G_v = \text{Gal}(\overline{F}_v/F_v)$ be the absolute Galois group of $F_v$, and let $I_v \leq G_v$ be the inertia group. Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{F}_v$ for all $v$, and hence embeddings $G_v \hookrightarrow G$. Let Frb$_v \in G_v$ be a lift of (arithmetic) Frobenius.

Let $\varepsilon_\ell : G_F \rightarrow \mathcal{O}_v^\times$ be the cyclotomic character (given by $\sigma(\zeta) = \zeta^{\varepsilon_\ell(\sigma)}$ for any $\sigma \in G_F$ and $\zeta \in \mu_{\ell^\infty}$), and let $\varepsilon_\ell : G_F \rightarrow \mathbb{F}_\ell^\times$ be its mod $\ell$ reduction.

Now take a maximal ideal $m \subseteq T^D(K)$, and note that $T^D(K)/m$ is a finite extension of $\mathbb{F}$.

It is well known (see [Car86]) that the ideal $m$ corresponds to a two-dimensional semisimple Galois representation $\overline{\rho}_m : G_F \rightarrow \text{GL}_2(T^D(K)/m) \subseteq \text{GL}_2(\mathbb{F}_\ell)$ satisfying:

1. $\overline{\rho}_m$ is odd.
2. If $v \nmid \mathfrak{D}, \ell, N_K$, then $\overline{\rho}_m$ is unramified at $v$ and we have
   \[\text{tr}(\overline{\rho}_m(\text{Frob}_v)) \equiv T_v \pmod{m}\]
   \[\text{det}(\overline{\rho}_m(\text{Frob}_v)) \equiv \text{Nm}(v)S_v \pmod{m}.\]
3. If $v|\ell$ and $v \nmid \mathfrak{D}, N_K$, then $\overline{\rho}_m$ is finite flat at $v$.
4. If $v|\mathfrak{D}$ then
   \[\overline{\rho}_m|_{G_v} \sim \left(\begin{array}{cc} \chi \varepsilon_\ell & * \\ 0 & \chi \end{array}\right),\]
   where $\chi : G_v \rightarrow \mathbb{F}_\ell^\times$ is an unramified character.

In keeping with property (4) above, for any $\mathcal{O}$-algebra $A$ we will say that a local representation $r : G_v \rightarrow \text{GL}_2(A)$ is Steinberg if it can be written (in some basis) as

\[r = \left(\begin{array}{cc} \chi \varepsilon_\ell & * \\ 0 & \chi \end{array}\right)\]

for some unramified character $\chi : G_v \rightarrow A^\times$. We say that a global representation $r : G_F \rightarrow \text{GL}_2(A)$ is Steinberg at $v$ if $r|_{G_v}$ is Steinberg.

Now if $\psi : G_F \rightarrow \mathcal{O}_v^\times$ is a character for which $\psi \varepsilon_\ell^{-1}$ has finite image, define the fixed determinant Hecke algebra $T^D(K)$ to be the maximal reduced $\ell$-torsion free quotient of $T^D(K)$ on which $\text{Nm}(v)S_v = \psi(\text{Frob}_v)$ for all $v \nmid \mathfrak{D}, \ell$ at which $K$ is unramified.

Note that by Chebotarev density, a maximal ideal $m \subseteq T^D(K)$ is in the support of $T^D(K)$ if and only if $\overline{\rho}_m$ has a lift $\rho : G_F \rightarrow \text{GL}_2(\mathcal{O})$ which is modular of level $K$ with $\det \rho = \psi$ (which in particular implies that $\det \overline{\rho}_m \equiv \psi \pmod{\lambda}$).
Now for any continuous absolutely irreducible representation $\rho : G_F \to \text{GL}_2(\mathbb{F}_\ell)$, define:

$$\mathcal{K}^D(\rho) = \left\{ K \subseteq D^{\times}(\mathbb{A}_F) \middle| \rho \sim \rho_m \text{ for some } m \subseteq T^D(K) \right\}$$

(that is, $\mathcal{K}^D(\rho)$ is the set of levels $K$ at which the representation $\rho$ can occur.)

Note that if $\mathcal{K}^D(\rho)$ is nonempty, then it has the form $\{K | K \subseteq K_{\text{min}}\}$ for some $m \subseteq T^D(K)$, called the minimal level of $\rho$. Given $K = \prod_{v \subseteq \mathcal{O}_F} K_v \in \mathcal{K}^D(\rho)$, we say that $K$ is of minimal level at some $v \subseteq \mathcal{O}_F$ if $K_v = K_{\text{min}}^v$.

From now on, fix an absolutely irreducible Galois representation $\rho : G_F \to \text{GL}_2(\mathbb{F}_\ell)$ for which $\mathcal{K}^D(\rho) \neq \emptyset$ (i.e. $\rho$ is “automorphic for $D$”). In particular, this implies that $\rho$ is odd, and satisfies the numbered conditions in Section 1.1.

Now given $K \in \mathcal{K}^D(\rho)$ and $m \subseteq T^D(K)$ for which $\rho \sim \rho_m$ we define the number:

$$\nu_{\rho}(K) := \begin{cases} \dim_{T^D(K)/m} S^D(K)[m] & \text{if } D \text{ is totally definite} \\ \frac{1}{2} \dim_{T^D(K)/m} S^D(K)[m] & \text{if } D \text{ is indefinite} \end{cases}$$

called the multiplicity of $\rho$ at level $K$. This number is closely related to the mod $\ell$ local-global compatibility conjectures given in [BDJ10]. Note that $\nu_{\rho}(K)$ does not depend on the choice of coefficient ring $\mathcal{O}$.

Theorem 1.1 is precisely the assertion that $\nu_{\rho}(K_{\text{min}}) = 2^k$.

We close this section by stating another result of our work:

**Theorem 1.2.** Let $\rho$ satisfy the conditions of Theorem 1.1. If $D$ to totally definite then trace map $S^D(K_{\text{min}})_m \otimes_{T^D(K_{\text{min}})_m} S^D(K_{\text{min}})_m \to \omega_{T^D(K_{\text{min}})_m}$, induced by the self-duality of $S^D(K_{\text{min}})_m$ is surjective (where $\omega_{T^D(K_{\text{min}})_m}$ is the dualizing sheaf of $T^D(K_{\text{min}})_m$), and moreover the natural map $T^D(K_{\text{min}})_m \to \text{End}_{T^D(K_{\text{min}})_m}(S^D(K_{\text{min}})_m)$ is an isomorphism. If $D$ is indefinite, then the natural map $T^D(K_{\text{min}})_m \to \text{End}_{T^D(K_{\text{min}})_m}[G_F](S^D(K_{\text{min}})_m)$ is an isomorphism.

As explained in [Eme02], this statement has applications towards the integral Eichler basis problem, so can likely be used to strengthen the results of Emerton [Eme02].

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2 Galois Deformation Rings

In this section we will define the various Galois deformation rings which we will consider in the rest of the paper, and review their relevant properties.

2.1 Local Deformation Rings

Fix a finite place $v$ of $F$ and a representation $\tau : G_v \to GL_2(\mathbb{F})$.

Let $\mathcal{C}_\mathcal{O}$ (resp. $C_{\mathcal{O}}^\wedge$) be the category of Artinian (resp. complete Noetherian) local $\mathcal{O}$-algebras with residue field $\mathbb{F}$. Consider the (framed) deformation functor $D^\square(\tau) : C_\mathcal{O} \to \text{Nm}(v)$ defined by

$$D^\square(\tau)(A) = \{ r : G_v \to GL_2(A), \text{ continuous lift of } \tau \}$$

$$= \{ (M, r, e_1, e_2) | M \text{ is a free rank } 2 \text{ } A\text{-module with a basis } (e_1, e_2) \text{ and } r : G_v \to \text{Aut}_A(M) \text{ such that the induced map } G \to \text{Aut}_A(M) = GL_2(A) \to GL_2(\mathbb{F}) \text{ is } \tau \}$$

It is well-known that this functor is pro-representable by some $R^\square(\tau) \in C_{\mathcal{O}}^\wedge$, in the sense that $D^\square(\tau) \equiv \text{Hom}_\mathcal{O}(R^\square(\tau), -)$. Furthermore, $\tau$ admits a universal lift $r^\square : G_v \to GL_2(R^\square(\tau))$.

For any continuous homomorphism, $x : R^\square(\tau) \to \mathcal{E}$, we obtain a Galois representation $r_x : G_v \to GL_2(\mathcal{E})$ lifting $\tau$, from the composition $G_v \xrightarrow{r^\square} GL_2(R^\square(\tau)) \xrightarrow{x} GL_2(\mathcal{E})$.

Now for any character $\psi : G_v \to \mathcal{O}^\times$ with $\psi \equiv \det \tau \pmod{\lambda}$ define $R^\square,\psi(\tau)$ to be the quotient of $R^\square(\tau)$ on which $\det r^\square(g) = \psi(g)$ for all $g \in G_v$. Equivalently, $R^\square,\psi(\tau)$ is the ring pro-representing the functor of deformations of $\tau$ with determinant $\psi$.

Given any two characters $\psi_1, \psi_2 : G_v \to \mathcal{O}^\times$ with $\det \tau \equiv \psi_1 \equiv \psi_2 \pmod{\lambda}$, and so (as $1+\lambda\mathcal{O}$ is pro-$\ell$ and $\ell \neq 2$) there is a unique $\chi : G_v \to \mathcal{O}^\times$ with $\psi_1 = \psi_2^\chi$.

But now the map $r \mapsto r \otimes \chi$ is an automorphism of the functor $D^\square(\tau)$ which can be shown to induce a natural isomorphism $R^\square,\psi_1(\tau) \cong R^\square,\psi_2(\tau)$. Thus, up to isomorphism, the ring $R^\square,\psi(\tau)$ does not depend choice of $\psi$.

We call $R^\square(\tau)$ (respectively $R^\square,\psi(\tau)$) the deformation ring (respectively the fixed determinant deformation ring) of $\tau$.

In order to prove our main results, we will also need to consider various deformation rings with fixed type. Instead of defining these in general, we will consider only the specific examples which will appear in our arguments.

If $v|\ell$ and $\tau$ and $\psi$ are both flat, define $R^{\square,\text{fl},\psi}(\tau)$ to be the ring pro-representing the functor of (framed) flat deformations of $\tau$ with determinant $\psi$. We will refrain from giving a precise definition of this, as it is not relevant to our discussion. We will refer the reader to [Kis09b], [FL82], [Ram93] and [CHT08] for more details, and use only the following result from [CHT08, Section 2.4]:

**Proposition 2.1.** If $F_v/\mathbb{Q}_\ell$ is unramified, then $R^{\square,\text{fl},\psi}(\tau) \cong \mathcal{O}[[X_1, \ldots, X_{d+[F_v:Q_\ell]]}]$. 

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Also if \( v \nmid \ell \), let \( R^{\Box, \min, \psi}(\tau) \) be the maximal reduced \( \lambda \)-torsion free quotient of \( R^{\Box, \psi}(\tau) \) with the following property: If \( x : R^{\Box, \min, \psi}(\tau) \to \overline{E} \) is a continuous homomorphism, then the corresponding lift \( r_x : G_v \to \mathrm{GL}_2(\overline{E}) \) of \( \tau \) has minimal level among all lifts of \( \tau \) with determinant \( \psi \). Again, we will refrain from giving a more detailed description of this, and instead we will use only the following well-known result (cf [Sho16, CHT08]):

**Proposition 2.2.** \( R^{\Box, \min, \psi}(\tau) \cong \mathcal{O}[[X_1, X_2, X_3]] \).

Now assume that \( v \nmid \ell \) and \( \tau \) is Steinberg (in the sense of Section 1.1). We define \( R^{\Box, \text{st}}(\tau) \) (called the Steinberg deformation ring) to be the maximal reduced \( \lambda \)-torsion free quotient of \( R^{\Box}(\tau) \) for which \( r_x : G_v \to \mathrm{GL}_2(\overline{E}) \) is Steinberg for every continuous homomorphism \( x : R^{\Box, \text{st}}(\tau) \to \overline{E} \).

Similarly if \( \psi : G_v \to \mathcal{O}^\times \) is an unramified character with \( \psi \equiv \det \tau \pmod{\lambda} \) (by assumption, \( \tau \) is Steinberg, and hence \( \det \tau \) is unramified), we define \( R^{\Box, \text{st}, \psi}(\tau) \) (called the fixed determinant Steinberg deformation ring) to be the maximal reduced \( \lambda \)-torsion free quotient of \( R^{\Box, \psi}(\tau) \) for which \( r_x : G_v \to \mathrm{GL}_2(\overline{E}) \) is Steinberg for every continuous homomorphism \( x : R^{\Box, \text{st}, \psi}(\tau) \to \overline{E} \).

It follows from our definitions that \( R^{\Box, \text{st}, \psi}(\tau) \) is the maximal reduced \( \lambda \)-torsion free quotient of \( R^{\Box, \text{st}}(\tau) \) on which \( \det \rho^{\Box}(g) = \psi(g) \) for all \( g \in G_v \).

### 2.2 Global Deformation Rings

Now take a representation \( \overline{\rho} : G_F \to \mathrm{GL}_2(\overline{F}) \) satisfying:

1. \( \overline{\rho} \) is absolutely irreducible.
2. \( \overline{\rho} \) is odd.
3. For each \( v|\ell \), \( \overline{\rho}|_{G_v} \) is finite flat.
4. For each \( v|\mathfrak{D}, \overline{\rho} \) is Steinberg at \( v \).
5. \( K^D(\overline{\rho}) \neq \emptyset \).

Let \( \Sigma^D_\ell \) be a set of finite places of \( F \) containing:

- All places \( v \) at which \( \overline{\rho} \) is ramified
- All places \( v|\mathfrak{D} \) (i.e. places at which \( D \) is ramified)
- All places \( v|\ell \)

(we allow \( \Sigma^D_\ell \) to contain some other places in addition to these), and let \( \Sigma \subseteq \Sigma^D_\ell \) consist of those \( v \in \Sigma^D_\ell \) with \( v \nmid \ell \), \( \mathfrak{D} \).

Now as in [Kis09b] define \( D^{\Box}_{F,S}(\overline{\rho}) \) (where \( \Sigma^D_\ell \subseteq S \)) to be the \( \mathcal{O} \)-algebra pro-representing the functor \( D^{\Box}_{F,S}(\overline{\rho}) : \mathcal{C}_\mathcal{O} \to \text{Nm}(v) \) which sends \( A \) to the set tuples \( (\rho : G_{F,S} \to \text{End}_A(M), \{ (e^v_1, e^v_2) \}_{v \in \Sigma^D_\ell}) \), where \( M \) is a free rank 2 \( A \)-module with an identification \( M/m_A = \mathbb{F}^2 \) sending \( \rho \) to \( \overline{\rho} \), and for each \( v \in \Sigma^D_\ell \), \( (e^v_1, e^v_2) \) is a basis for \( M \), lifting the standard basis for \( M/m_A = \mathbb{F}^2 \), up to equivalence.

Also define the unframed deformation ring \( R_{F,S}(\overline{\rho}) \) to be the \( \mathcal{O} \)-algebra pro-representing the functor \( D_{F,S}(\overline{\rho}) : \mathcal{C}_\mathcal{O} \to \text{Nm}(v) \) which sends \( A \) to the set of free rank 2 \( A \) modules \( M \) with action \( \rho : G_{F,S} \to \)}
End_A(M) for which $\rho \equiv \overline{\rho} \pmod{m_A}$, up to equivalence. This exists because $\overline{\rho}$ is absolutely irreducible. We will let $\rho^{\text{univ}} : G_F \to \GL_2(R_{F,S}(\overline{\rho}))$ denote the universal lift of $\overline{\rho}$.

Now take any character $\psi : G_F \to \mathcal{O}^\times$ for which:

1. $\psi \equiv \det \overline{\rho} \pmod{\lambda}$.
2. $\psi$ is unramified at all places outside of $\Sigma^D_\ell$, and all places dividing $\mathfrak{D}$.
3. $\psi$ is flat at all places dividing $\ell$.
4. $\psi e_{\ell}^{-1}$ has finite image.

Note that as $\psi e_{\ell}^{-1}$ has finite image, condition (3) is equivalent to the assertion that $\psi e_{\ell}^{-1}$ is unramified at all places dividing $\ell$.

Let $\mathcal{D}^{\square,\psi}_{F,S} \subseteq \mathcal{D}^{\square}_{F,S}$ be the subfunctor of $\mathcal{D}^{\square}_{F,S}$ which sends $A$ to the set of tuples $(\rho : G_{F,S} \to \End_A(M), \{ (e^\rho_1, e^\rho_2) \}_{v \in \Sigma^D_\ell})$ (up to equivalence) in $\mathcal{D}^{\square}_{F,S}(A)$ for which $\det \rho = \psi$. Define $\mathcal{D}^{\psi}_{F,S} \subseteq \mathcal{D}^{\square}_{F,S}$ similarly. Let $R^{\square,\psi}_{F,S}(\overline{\rho})$ and $R^{\psi}_{F,S}(\overline{\rho})$ to be the rings pro-representing $\mathcal{D}^{\square,\psi}_{F,S}$ and $\mathcal{D}^{\psi}_{F,S}$. Equivalently, these are the quotients of $R^{\square,\psi}_{F,S}(\overline{\rho})$ and $R_{F,S}(\overline{\rho})$, respectively, on which $\det \rho = \psi$.

Now note that the morphism of functors

$$
(\rho, \{ (e^\rho_1, e^\rho_2) \}_{v \in \Sigma^D_\ell}) \mapsto (\rho|_{G_v} : G_v \to \End_A(M), (e^\rho_1, e^\rho_2))_{v \in \Sigma^D_\ell}
$$

induces a map:

$$
\pi : R_{\text{loc}} := \bigotimes_{v \in \Sigma^D_\ell} R^{\square}(\overline{\rho}|_{G_v}) \to R^{\square}_{F,S}.
$$

Now consider the ring

$$
R^{\square,\mathfrak{D},\ell}_{\Sigma,\mathfrak{D},\ell} := \left[ \bigotimes_{v \in \Sigma} R^{\square,\mathfrak{D},v}_v(\overline{\rho}|_{G_v}) \right] \otimes \left[ \bigotimes_{v \in \mathfrak{D}} R^{\square,\mathfrak{D},v}_v(\overline{\rho}|_{G_v}) \right] \otimes \left[ \bigotimes_{v \in \mathfrak{D}} R^{\square,\mathfrak{D},v}_v(\overline{\rho}|_{G_v}) \right],
$$

so that $R^{\square,\mathfrak{D},\ell}_{\Sigma,\mathfrak{D},\ell}$ is a quotient of $R_{\text{loc}}$. Using the map $\pi$ above, we may now define $R^{\square,\mathfrak{D}}_{F,S}(\overline{\rho}) := R^{\square,\mathfrak{D}}_{F,S} \otimes_{R_{\text{loc}}} R^{\square,\mathfrak{D},\ell}_{\Sigma,\mathfrak{D},\ell}$. We may also define $R^{\psi}_{\text{loc}}, R^{\psi}_{\Sigma,\mathfrak{D},\ell}$ and $R^{\square,\mathfrak{D},\psi}_{F,S}(\overline{\rho})$ analogously, by adding superscripts of $\psi$ to all of the rings used in the definitions.

Also note that the morphism of functors $(\rho, \{ (e^\rho_1, e^\rho_2) \}_{v \in \Sigma^D_\ell}) \mapsto \rho$ induces a map $R^{\psi}_{F,S}(\overline{\rho}) \to R^{\square,\psi}_{F,S}(\overline{\rho})$. As in [Kis09b, (3.4.11)] this maps is formally smooth of dimension $j := 4|\Sigma^D_\ell| - 1$, and so we may identify $R^{\square,\psi}_{F,S}(\overline{\rho}) = R^{\psi}_{F,S}(\overline{\rho})[[w_1, \ldots, w_j]]$.

We can now define a map $R^{\square,\psi}_{F,S}(\overline{\rho}) \to R^{\psi}_{F,S}(\overline{\rho})$ by sending each $w_i$ to 0. Using this map, we may now define a unframed version of $R^{\square,\mathfrak{D},\psi}_{F,S}(\overline{\rho})$ via

$$
R^{\square,\psi}_{F,S}(\overline{\rho}) := R^{\square,\mathfrak{D},\psi}_{F,S}(\overline{\rho}) \otimes_{R^{\square,\psi}_{F,S}(\overline{\rho})} R^{\psi}_{F,S}(\overline{\rho}) = R^{\square,\psi}_{F,S}(\overline{\rho})/\langle w_1, \ldots, w_j \rangle. \quad (3)
$$

This definition may seem somewhat strange and ad-hoc. We have defined it this way because it will be convenient for our argument to view $R^{\mathfrak{D},\psi}_{F,S}(\overline{\rho})$ as a quotient of $R^{\psi}_{F,S}(\rho)$. Fortunately we will see later (in the remark following...
It follows from these definitions that the maps \( x : R^{\psi}_{F,S}(\mathfrak{p}) \to \mathcal{E} \) that factor through \( R^{D,\psi}_{F,S}(\mathfrak{p}) \) are precisely those for which the induced representation \( \rho_x : G_F \to GL_2(R^{\psi}_{F,S}(\mathfrak{p})) \to GL_2(\mathcal{E}) \) satisfies:

- \( \overline{\rho_x}|_{G_v} \) is flat at all \( v|\ell \)
- \( \overline{\rho_x}|_{G_v} \) is Steinberg at all \( v|\mathcal{D} \)
- \( \overline{\rho_x}|_{G_v} \) has minimal level at all \( v \in \Sigma \).

(This is simply because the definition of \( R^{D,\psi}_{F,S}(\mathfrak{p}) \) is such that any map \( x : R^{\psi}_{F,S}(\mathfrak{p}) \to \mathcal{E} \) factors through \( R^{D,\psi}_{F,S}(\mathfrak{p}) \) if and only if the corresponding map \( x : R^{\psi}_{F,S} \to R^{\psi}_{F,S}(\mathfrak{p}) \to \mathcal{E} \) factors through \( R^{D,\psi}_{F,S}(\mathfrak{p}) \).

In order to prove Theorem 1.2 we will need slightly more refined information about the relationship between \( R^{\psi}_{F,S}(\mathfrak{p}) \) and \( R^{\psi}_{F,S}(\mathfrak{p}) \).

Let \( \overline{\psi} = \det \mathfrak{p} : G_F \to \mathbb{F}^\times \) be the reduction of \( \psi \). Let \( \mathcal{D}_{F,S}(\overline{\psi}) : \mathcal{C}_O \to \text{Set} \) be the functor which sends \( A \in \mathcal{C}_O \) to the set of maps \( \chi : G_{F,S} \to A^\times \) satisfying \( \chi \equiv \overline{\psi} \pmod{m_A} \), up to equivalence. Let \( R_{F,S}(\overline{\psi}) \) be the ring pro-representing \( \mathcal{D}_{F,S}(\overline{\psi}) \).

Now for any \( A \in \mathcal{C}_O \), \( A \) is a finite ring of \( \ell \)-power order, and so \( m_A \subseteq A \) also has \( \ell \)-power order. It follows that \( (1 + m_A, x) \) is an abelian multiplicative group of \( \ell \)-power order. In particular, as \( \ell \) is odd, the map \( x \mapsto x^2 \) is an automorphism of \( (1 + m_A, x) \), and hence it has an inverse \( \sqrt{\cdot} : (1 + m_A, x) \to (1 + m_A, x) \). It is easy to see that \( x \mapsto x^2 \), and hence \( x \mapsto \sqrt{x} \), commutes with morphisms in \( \mathcal{C}_O \), and is thus an automorphism of the functor \( A \to (1 + m_A, x) \) from \( \mathcal{C}_O \) to \( \text{Ab} \).

Now consider any \( \rho : G_{F,S} \to \text{End}(M) \) in \( \mathcal{D}_{F,S}(\overline{\mathfrak{p}})(A) \). By definition we have \( \det \rho \equiv \det \overline{\mathfrak{p}} \equiv \psi \pmod{m_A} \), and so \( (\det \rho)^{-1}\psi \equiv 1 \pmod{m_A} \). That is, the image of \( (\det \rho)^{-1}\psi : G_{F,S} \to A^\times \) lands in \( 1 + m_A \). By the above work, it follows that there is a unique character \( \sqrt{(\det \rho)^{-1}\psi} : G_{F,S} \to 1 + m_A \subseteq A^\times \) with \( (\sqrt{(\det \rho)^{-1}\psi})^2 = (\det \rho)^{-1}\psi \). Thus we may define a representation \( \rho^\psi : G_{F,S} \to \text{End}(M) \) by \( \rho^\psi := (\sqrt{(\det \rho)^{-1}\psi})\rho \). Notice that \( \rho^\psi \in \mathcal{D}_{F,S}(\overline{\mathfrak{p}})(A) \) and we have \( \det \rho^\psi = \psi \), so that \( \rho^\psi \in \mathcal{D}_{F,S}(\overline{\mathfrak{p}})(A) \). It is easy to see that the map \( \rho \mapsto \rho^\psi \) is a natural transformation \( \mathcal{D}_{F,S} \to \mathcal{D}_{F,S}^\psi \).

We now claim that the map \( \mathcal{D}_{F,S}(\overline{\mathfrak{p}}) \to \mathcal{D}_{F,S}(\overline{\psi}) \times \mathcal{D}_{F,S}(\overline{\mathfrak{p}}) \) given by \( \rho \mapsto (\det \rho, \rho^\psi) \) is an isomorphism of functors. Indeed, it has an inverse given by \( (\chi, \rho) \mapsto \sqrt{\chi \psi^{-1}} \rho \). Looking at the rings these functors represent gives the following:

**Lemma 2.3.** There is a natural isomorphism \( R_{F,S}(\overline{\psi}) \otimes \mathcal{O} R^{\psi}_{F,S}(\mathfrak{p}) \xrightarrow{\sim} R_{F,S}(\mathfrak{p}) \) of \( \mathcal{O} \)-algebras, induced by the natural transformation \( \rho \mapsto (\det \rho, \rho^\psi) \).

Lemma 2.3 may be though of as giving a natural way of separating the determinant of a representation \( \rho : G_{F,S} \to GL_2(A) \) from the rest of the representation.

---

Lemma 4.9) that \( R^{D,\psi}_{F,S}(\mathfrak{p}) \) is actually a fairly nicely behaved ring in the cases relevant to us, and so this definition will likely coincide with most “natural” ones.
2.3 Two Lemmas about Deformation Rings

We finish this section by stating two standard results (cf. [Kis09b]) which will be essential for our discussion of Taylor–Wiles–Kisin patching in Section 4.

The first concerns the existence of an “$R \to T$” map:

**Lemma 2.4.** Assume that $\bar{\rho}$ satisfies all of the numbered conditions listed in Section 2.2. Take $K \in \mathcal{K}^D(\bar{\rho})$ and let $S$ be a set of finite places of $F$ containing $\Sigma^D_\ell$ such that $K$ is unramified outside of $S$. Then there is a surjective map $R_{F,S}(\bar{\rho}) \to T^D(K)$, which induces a surjective map $R_{\psi,F,S}(\bar{\rho}) \to T^D_{\psi}(K)$ for any character $\psi : G_F \to \mathbb{O}^\times$ lifting $\det \rho$. This map factors through $R_{\psi,F,S}(\bar{\rho}) \to R_{\psi,F,S}(\bar{\rho})$.

The second concerns the existence of “Taylor–Wiles” primes:

**Lemma 2.5.** Assume that $\bar{\rho}$ satisfies all of the numbered conditions listed in Section 2.2 and condition (4) of Theorem 1.1. Let $S$ be a set of finite places of $F$ containing $\Sigma^D_\ell$, such for any prime $v \in S \setminus \Sigma^D_\ell$, $\text{Nm}(v) \not\equiv 1 \pmod{\ell}$ and the ratio of the eigenvalues of $\bar{\rho} (\text{Frob}_v)$ is not equal to $\text{Nm}(v) \pm 1$ in $\mathbb{F}^\times$.

Then there exist integers $r, g \geq 1$ such that for any $n \geq 1$, there is a finite set $Q_n$ of primes of $F$ for which:

- $Q_n \cap S = \emptyset$.
- $\# Q_n = r$.
- For any $v \in Q_n$, $\text{Nm}(v) \equiv 1 \pmod{\ell^n}$.
- For any $v \in Q_n$, $\bar{\rho} (\text{Frob}_v)$ has distinct eigenvalues.
- There is a surjection $R_{\Sigma, D, \ell, \psi}^\square[[x_1, \ldots, x_g]] \to R_{F, S \cup Q_n, \psi}^\square D(\bar{\rho})$.

Moreover, we have $\dim R_{\Sigma, D, \ell, \psi}^\square = r + j - g + 1$.

From now on we will write $R_\infty$ to denote $R_{\Sigma, D, \ell, \psi}^\square[[x_1, \ldots, x_g]]$ so that $\dim R_\infty = r + j + 1$. By the results of Section 2.1 we have

$$R_\infty = \bigotimes_{v | D} \hat{R}_v^{\square, \psi, \bar{\rho}([G_v])}[[x_1, \ldots, x_{g'}]]$$

for some integer $g'$. In Section 3 below, we will use the results of [Sho16] to explicitly compute the ring $R_\infty$, and then use the theory of toric varieties to study modules over $R_\infty$.

In Chapter 4, we will use Lemma 2.4 and 2.5 to construct a particular module $M_\infty$ over $R_\infty$ out of a system of modules over the rings $T^D(K)$, and then use the results of Chapter 3 to deduce the structure of $M_\infty$. This will allow us to prove Theorems 1.1 and 1.2.
3 Class Groups of Local Deformation Rings

In our situation, all of the local deformation rings which will be relevant to us were computed in [Sho16]. In this section, we will use this description to explicitly describe the ring $R_\infty$, and to study its class group (or rather, the class group of a related ring).

We first introduce some notation which we will use for the rest of this paper. If $R$ is any Noetherian local ring, we will always use $m_R$ to denote its maximal ideal.

If $M$ is a (not necessarily free) finitely generated $R$-module, we will say that the rank of $M$, denoted by $\operatorname{rank}_R M$ is the cardinality of its minimal generating set.

If $R$ is a domain we will write $K(R)$ for its fraction field. If $M$ is a finitely generated $R$-module, then we will say that the generic rank of $M$, denoted $\operatorname{g.rank}_R M$ is the $K(R)$-dimension of $M \otimes_R K(R)$ (that is, the rank of $M$ at the generic point of $R$).

Also if $R$ is a (not necessarily local) any Cohen–Macaulay ring with a dualizing sheaf, we will use $\omega_R$ to denote the dualizing sheaf of $R$.

For any finitely generated $R$-module $M$, we will let $M^* = \operatorname{Hom}_R(M, \omega_R)$. We say that $M$ is reflexive if the natural map $^5 M \to M^{**}$ is an isomorphism.

We will let $\operatorname{Cl}(R)$ denote the Weil divisor class group of $R$, which is isomorphic the group of generic rank 1 reflexive modules over $R$. For any generic rank 1 reflexive sheaf $M$, let $[M] \in \operatorname{Cl}(R)$ denote the corresponding element of the class group. The group operation is then defined by $[M] + [N] := [(M \otimes_R N)^{**}]$. Note that $[\omega_R] \in \operatorname{Cl}(R)$ and we have $[M^*] = [\omega_R] - [M]$ for any $[M] \in \operatorname{Cl}(R)$.

Lastly, given any reflexive module $M$, the natural perfect pairing $M^* \times M \to \omega_R$ gives rise to a natural map $\tau_M : M^* \otimes_R M \to \omega_R$ (defined by $\tau_M(\varphi \otimes x) = \varphi(x)$) called the trace map.

Also we will let

$$k := \# \left\{ v \mid \mathfrak{p} \text{ is unramified at } v \text{ and, } \mathfrak{p}(\text{Frob}_v) \text{ is a scalar} \right\}.$$ 

as in Theorem 1.1.

Our main result of this section is the following:

**Theorem 3.1.** If $M_\infty$ is a finitely-generated module over $R_\infty$ satisfying:

1. $M_\infty$ is maximal Cohen–Macaulay over $R_\infty$.
2. We have $M^*_\infty \cong M_\infty$ (and hence $M_\infty$ is reflexive).

---

4Which will be the case for all Cohen–Macaulay rings we will consider.

5As is it fairly easy to show that the dual of a finitely generated $R$-module is reflexive (cf [Sta18, Tag 0AV2]) this definition is equivalent to simply requiring that there is some isomorphism $M \xrightarrow{\sim} M^{**}$. In particular, if $M \cong M^*$ then $M$ is automatically reflexive.
3. g. rank\( R_\infty \cdot M_\infty = 1 \).

then \( \dim_F M_\infty / m_{R_\infty} = 2^k \). Moreover, the trace map \( \tau_{M_\infty} : M_\infty \otimes_{R_\infty} M_\infty \to \omega_{R_\infty} \) is surjective.

Thus, to prove Theorem 1.1, it will suffice to construct a module \( M_\infty \) over \( R_\infty \) satisfying the conditions of Theorem 3.1 with \( \dim_F M_\infty / m_{R_\infty} = \nu \rho \). Moreover, the trace map \( \tau_{M_\infty} : M_\infty \otimes_{R_\infty} M_\infty \to \omega_{R_\infty} \) is surjective.

Our primary strategy for proving Theorem 3.1 is to note that conditions (1) and (3) imply that \( M_\infty \) is the module corresponding to a Weil divisor on \( R_\infty \), and condition (2) implies that we have \( 2[M_\infty] = [\omega_{R_\infty}] \) in \( \text{Cl}(R_\infty) \). Provided that \( \text{Cl}(R_\infty) \) is 2-torsion free, this means that conditions (1), (2) and (3) uniquely characterize the module \( M_\infty \). Proving the theorem would thus simply be a matter of computing the unique module \( M_\infty \) satisfying the conditions of the theorem explicitly enough.

Unfortunately, while we can give a precise description of the ring \( R_\infty \) in our situation, it is difficult to directly compute \( \text{Cl}(R_\infty) \) from that description. Instead, we will first reduce the statement of Theorem 3.1 to a similar statement over the ring \( R_\infty := R_\infty / \lambda \) (see Theorems 3.3 and 3.5 below). We will then be able to directly compute \( \text{Cl}(R) \), and the unique module \( M \) with \( 2[M] = [\omega_R] \) in \( \text{Cl}(R) \), by using the theory of toric varieties.

In Section 3.1 we summarize the computations in [Sho16] to explicitly describe the rings \( R_\infty \) and \( \overline{\text{Cl}}(R_\infty) \), and reduce Theorem 3.1 to the corresponding statement over \( \overline{\text{Cl}}(R_\infty) \) (Theorem 3.3). In Section 3.2 we introduce the ring \( R \), and show that it is the coordinate ring of an affine toric variety. Finally in Section 3.3 we use the theory of toric varieties to compute \( \text{Cl}(R) \), which allows us to prove a “de-completed” mod \( \lambda \) version of Theorem 3.1. In Section 3.4 we adapt the method of Danilov [Dan68] for computing the class groups of completions of graded rings to show that \( \text{Cl}(R) \cong \text{Cl}(\overline{R_\infty}) \), from which we deduce Theorem 3.3 and hence Theorem 3.1.

### 3.1 Explicit Calculations of Local Deformation Rings

In order to prove Theorem 3.1, it will be necessary to first compute the ring \( R_\infty \), or equivalently to compute \( R^{\square, st, \psi}(p|G_v) \) for all \( v|\mathcal{D} \).

These computations were essentially done by Shotton [Sho16], except that he considers the non fixed determinant version, \( R^{\square, \psi}(p|G_v) \) instead of \( R^{\square, st, \psi}(p|G_v) \). Fortunately, it is fairly straightforward to recover \( R^{\square, \psi}(p|G_v) \) from \( R^{\square, st, \psi}(p|G_v) \). Specifically, we get:

**Theorem 3.2.** Take any place \( v|\mathcal{D} \). Recall that we have assumed that \( \text{Nm}(v) \not\equiv -1 \pmod{\ell} \). If the residual representation \( p|G_v : G_v \to \text{GL}_2(F) \) is not scalar, then \( R^{\square, st, \psi}(p|G_v) \cong \mathcal{O}[[X_1, X_2, X_3]] \).

If \( p|G_v : G_v \to \text{GL}_2(F) \) is scalar (which can only happen when \( \text{Nm}(v) \equiv 1 \pmod{\ell} \)) then

\[
R^{\square, st, \psi}(p|G_v) \cong S_v := \mathcal{O}[[A, B, C, X, Y, Z]]/\mathcal{I}_v
\]
where $I_v$ is the ideal generated by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} A & B & X & Y \\ C & A & Z & X + 2 \text{Nm}(v)^{-1} \end{pmatrix}. $$

The ring $S_v$ is a Cohen–Macaulay and non-Gorenstein domain of relative dimension 3 over $O$. $(\lambda, C, Y, B - Z)$ is a regular sequence for $S_v$. Moreover, $S_v[1/\lambda]$ is formally smooth of dimension 3 over $E$.

**Proof.** For convenience, let $R_{st} = R_{st,\square}(P_{G_v})$ and $R^\psi_{st} = R_{st,\square,\psi}(P_{G_v})$. By definition, $R^\psi_{st}$ is the maximal reduced $\lambda$-torsion free quotient of $R_{st}$ on which $\det \rho^\square = \psi(g)$ for all $g \in G_v$.

Now let $I_v/\hat{P}_v \cong \mathbb{Z}_\ell$ be the maximal pro-$\ell$ quotient of $I_v$, so that $\hat{P}_v \subseteq G_v$, and $T_v := G_v/\hat{P}_v \cong \mathbb{Z}_\ell \times \mathbb{Z}$. Now let $\sigma, \phi \in T_v$ be topological generators for $\mathbb{Z}_\ell$ and $\mathbb{Z}$, respectively (chosen so that $\phi$ is a lift of arithmetic Frobenius, so that $\phi \sigma \phi^{-1} = \sigma \text{Nm}(v)$).

Now as in [Sho16], we may assume that the universal representation $\rho^\square : G_v \to \text{GL}_2(R_{st})$ factors through $T_v$. As we already have $\det \rho^\square(\sigma) = 1 = \psi(\sigma)$, it follows that $R^\psi_{st}$ is the maximal reduced $\lambda$-torsion free quotient of $R$ on which $\det \rho^\square(\phi) = \psi(\phi)$.

As explained in Section 2, up to isomorphism the ring $R^\psi_{st}$ is unaffected by the choice of $\psi$, so it will suffice to prove the claim for a particular choice of $\psi$. Thus from now on we will assume that $\psi$ is unramified and $\psi(\phi) = \frac{\text{Nm}(v)}{\text{Nm}(v) + 1} t^2$ where

$$t = \begin{cases} \text{Nm}(v) + 1 & \text{Nm}(v) \not\equiv \pm 1 \pmod{\ell} \\ 2 & \text{Nm}(v) \equiv 1 \pmod{\ell} \end{cases}$$

so that $t \equiv \text{Nm}(v) + 1 \equiv \text{tr} \bar{p}(\phi) \pmod{\ell}$ (this particular choice of $t$ is made to agree with the computations of [Sho16]).

But now by the definition of $R_{st} = R_{st,\square}(P_{G_v})$ we have that $\text{Nm}(v) (\text{tr} \rho^\square(\phi))^2 = (\text{Nm}(v) + 1)^2 \det \rho^\square(\phi)$ and so

$$\det \rho^\square(\phi) = \frac{\text{Nm}(v)}{(\text{Nm}(v) + 1)^2} (\text{tr} \rho^\square(\phi))^2$$

(where we have used the fact that $\text{Nm}(v) \not\equiv -1 \pmod{\ell}$, and so $\text{Nm}(v) + 1$ is a unit in $O$).

It follows that

$$\det \rho^\square(\phi) - \psi(\phi) = \frac{\text{Nm}(v)}{(\text{Nm}(v) + 1)^2} (\text{tr} \rho^\square(\phi)) - \frac{\text{Nm}(v)}{(\text{Nm}(v) + 1)^2} t^2 = \frac{\text{Nm}(v)}{(\text{Nm}(v) + 1)^2} (\text{tr} \rho^\square(\phi) + t)(\text{tr} \rho^\square(\phi) - t).$$

But now

$$\frac{\text{Nm}(v)}{(\text{Nm}(v) + 1)^2} (\text{tr} \rho^\square(\phi) + t) \equiv \frac{2\text{Nm}(v)}{(\text{Nm}(v) + 1)^2} \text{tr} \bar{p}(\phi) \equiv \frac{2\text{Nm}(v)}{\text{Nm}(v) + 1} \pmod{m_R}$$

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and so as $\ell \nmid 2, \Nm(v), \Nm(v) + 1$ we get that \( \frac{\Nm(v)}{(\Nm(v) + 1)^2} \left( \tr \rho^\sqcap(\phi) + t \right) \) is a unit in $R_{st}$. It follows that $R^\psi_{st}$ is the maximal reduced $\lambda$-torsion free quotient of

\[
R^\psi_{st,0} := \frac{R_{st}}{(\det \rho^\sqcap(\sigma) - \psi(\phi)) \tr \rho^\sqcap(\phi) - t} = \frac{R_{st}}{\tr \rho^\sqcap(\phi) - t}
\]

It now follows immediately from Shotton’s computations that in each case $R^\psi_{st,0}$ is already reduced and $\lambda$-torsion free (and so $R^\psi_{st} = R^\psi_{st,0}$) and has the form described in the statement of Theorem 3.2 above.

Indeed, first assume that $\Nm(v) \not\equiv \pm 1 \pmod{\ell}$. By [Sho16, Proposition 5.5] we may write $R_{st} = \O[[B,P,X,Y]]$ with

\[
\rho^\sqcap(\sigma) = \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x + B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix}
\]

\[
\rho^\sqcap(\phi) = \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix}^{-1} \begin{pmatrix} \Nm(v)(1 + P) & 0 \\ 0 & 1 + P \end{pmatrix} \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix},
\]

for some $x \in \O$. Thus we have

\[
\tr \rho^\sqcap(\phi) = (\Nm(v) + 1)(1 + P) = t + tP
\]

and so (as $t = \Nm(v) + 1 \in \O$ is a unit), $R^\psi_{st,0} = R_{st}/(tP) \cong \O[[B,P,X,Y]]/(P) \cong \O[[B,X,Y]]$, as desired.

Now assume that $\Nm(v) \equiv 1 \pmod{\ell}$. Again, following the computations of [Sho16, Proposition 5.8] we can write

\[
\rho^\sqcap(\sigma) = \begin{pmatrix} 1 + A & x + B \\ C & 1 - A \end{pmatrix}
\]

\[
\rho^\sqcap(\phi) = \begin{pmatrix} 1 + P & y + R \\ S & 1 + Q \end{pmatrix}
\]

(for $x, y \in \O$) where $A, B, C, P, Q, R$ and $S$ topologically generate $R_{st}$. Now following Shotton’s notation, let $T = P + Q$, so that $\tr \rho^\sqcap(\phi) = 2 + T = t + T$ and thus $R^\psi_{st,0} = R_{st}/(T)$. In both cases ($\mathfrak{g}_{G_v}$ non-scalar and scalar) Shotton’s computations immediately give the desired description of $R^\psi_{st}$.\(^6\)

Moreover, Shotton shows that $S_v$ is indeed Cohen–Macaulay and non Gorenstein of relative dimension 3 over $\O$, and that $S_v[1/\lambda]$ is formally smooth of dimension 3 over $E$. As $S_v$ is Cohen–Macaulay, the claim that $(\lambda,C,Y,B - Z)$ is a regular sequence simply follows by noting that

\[
S_v/(\lambda,C,Y,B - Z) \cong \F[[A,B,X]]/(A^2, AB, AX, B^2, BX, X^2) = \F \oplus \F A \oplus \F B \oplus \F X
\]

\(^6\)In Shotton’s notation, when $\mathfrak{g}_{G_v}$ is scalar $R^\psi_{st}$ would be cut out by the $2 \times 2$ minors of the matrix

\[
\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & -X_2 & Y_3 & 2^{2^{\Nm(v)+1}} \end{pmatrix}
\]

This is equivalent to the form stated in Theorem 3.2 via the variable substitutions $A = X_1, B = -X_2, C = Y_1, X = X_3, Y = X_4$ and $Z = Y_3$.\(^6\)
is a zero dimensional ring, and so \((\lambda, C, Y, B - Z)\) is a system of parameters. □

Thus letting \(D_1|\mathcal{D}\) be the product of the places \(v|\mathcal{D}\) at which \(\overline{p}|_{G_v} : G_v \to GL_2(\mathbb{F})\) is scalar, we have

\[
R_\infty = \bigotimes_{v|D_1} S_v \left[ [x_1, \ldots, x_s] \right]
\]

for some integer \(s\). As the rings \(S_v\) are all Cohen–Macaulay by Theorem 3.2, it follows that \(R_\infty\) is as well.

Now note that the description of \(S_v\) in Theorem 3.2 becomes much simpler if we work in characteristic \(\ell\). Indeed, if \(\overline{p}|_{G_v}\) is scalar then \(Nm(v) \equiv 1 \mod \ell\) and so, \(\overline{S} := S_v/\lambda\) is an explicit graded ring not depending on \(v\). Specifically, we have \(S = \mathbb{F}[A, B, C, X, Y, Z]/\mathcal{I}\) where \(\mathcal{I}\) is the (homogeneous) ideal generated by the \(2 \times 2\) minors of the matrix \(\begin{pmatrix} A & B & X & Y \\ C & A & Z & X \end{pmatrix}\).

It thus follows that

\[
\overline{R}_\infty := R_\infty/\lambda \cong \overline{S} \otimes_{k} [[x_1, \ldots, x_s]],
\]

which will be much easier to work with than \(R_\infty\). In particular, note that \(\overline{R}_\infty\) is still Cohen–Macaulay, as \(R_\infty\) is \(\lambda\)-torsion free by definition.

It will thus be useful to reduce Theorem 3.1 the following \textquotedblleft mod \(\lambda\"\textquotedblright case:

\textbf{Theorem 3.3.} If \(\overline{M}_\infty\) is a finitely-generated module over \(\overline{R}_\infty\) satisfying:

\begin{enumerate}
  \item \(\overline{M}_\infty\) is maximal Cohen–Macaulay over \(\overline{R}_\infty\)
  \item We have \(\overline{M}_\infty \cong \overline{M}_\infty\).
  \item \(g.\text{rank}_{\overline{R}_\infty} \overline{M}_\infty = 1\).
\end{enumerate}

then \(\dim_{\overline{R}_\infty} \overline{M}_\infty/m_{\overline{R}_\infty} = 2^k\). Moreover, the trace map \(\tau_{\overline{M}_\infty} : \overline{M}_\infty \otimes_{\overline{R}_\infty} \overline{M}_\infty \rightarrow \omega_{\overline{R}_\infty}\) is surjective.

\textit{Proof that Theorem 3.3 implies 3.1.} Assume that Theorem 3.3 holds, and that \(M_\infty\) satisfies the hypotheses of Theorem 3.1. As \(R_\infty\) is flat over \(O\), it is \(\lambda\)-torsion free and thus \(\lambda\) is not a zero divisor on \(M_\infty\) (by condition (1)). It follows that \(\overline{M}_\infty := M_\infty/\lambda\) is maximal Cohen–Macaulay over \(\overline{R}_\infty\) and \(g.\text{rank}_{\overline{R}_\infty} \overline{M}_\infty = g.\text{rank}_{\overline{R}_\infty} M_\infty = 1\). (In general, if \(R\) is Cohen–Macaulay and \(M\) is maximal Cohen–Macaulay over \(R\), then for any regular element \(x \in R\), \(g.\text{rank}_{R/x} (M/x) = g.\text{rank}_R M\), provided \(R/x\) is also a domain.)

Moreover, as \(M_\infty\) and \(\omega_{R_\infty}\) are both flat over \(O\), we have that

\[
\overline{M}_\infty = M_\infty/\lambda = \text{Hom}_{R_\infty}(M_\infty, \omega_{R_\infty})/\lambda = \text{Hom}_{R_\infty}(M_\infty/\lambda, \omega_{R_\infty}/\lambda) = \text{Hom}_{\overline{R}_\infty}(\overline{M}_\infty, \omega_{\overline{R}_\infty}),
\]

where we have used the fact that \(\omega_{R_\infty}/\lambda \cong \omega_{\overline{R}_\infty}\), by [Eis95, Chapter 21.3]. Thus \(\overline{M}_\infty\) is self-dual. Thus \(\overline{M}_\infty\) satisfies all of the hypotheses of Theorem 3.3, and so \(\dim_{\overline{R}_\infty} \overline{M}_\infty/m_{\overline{R}_\infty} = 2^k\) and \(\tau_{\overline{M}_\infty}\) is surjective.
Now we obviously have that $\overline{M}_\infty/m_{R_\infty} \cong M_\infty/m_{R_\infty}$, so the first conclusion of Theorem 3.1 follows.

Also, the trace map $\tau_{M_\infty} : M_\infty \otimes_{R_\infty} M_\infty \to \omega_{R_\infty}$ is just the mod-$\lambda$ reduction of the map $\tau_{M_\infty} : M_\infty \otimes M_\infty \to \omega_{R_\infty}$, so it follows that $\tau_{M_\infty}$ is surjective if and only if $\tau_{M_\infty}$ is. Thus the second conclusion of Theorem 3.1 follows.

As hinted above, we will prove Theorem 3.3 by computing the class group of $R_\infty$.

We finish this section by proving the following lemma, which will make the second conclusion of Theorem 3.3 easier to prove (and will also be useful in the proof of Theorem 1.2):

**Lemma 3.4.** If $R$ is a Cohen–Macaulay ring with a dualizing sheaf $\omega_R$, and $M$ is a reflexive $R$-module, then the trace map $\tau_M : M^* \otimes R M \to \omega_R$ is surjective if and only if there exists an $R$-module surjection $M^* \otimes R M \twoheadrightarrow \omega_R$. 

**Proof.** Assume that $f : M^* \otimes R M \to \omega_R$ is a surjection. Take any $\alpha \in \omega_R$. Then we can write

$$\alpha = f \left( \sum_{i \in I} b_i \otimes c_i \right) = \sum_{i \in I} f(b_i \otimes c_i)$$

for some finite index set $I$ and some $b_i \in M^*$ and $c_i \in M$. For each $i \in I$, consider the $R$-linear map $\varphi_i : M \to \omega_R$ defined by $\varphi_i(c) = f(b_i \otimes c)$. Then we have $\varphi_i \in M^*$ for all $i$ and so

$$\alpha = \sum_{i \in I} f(b_i \otimes c_i) = \sum_{i \in I} \varphi_i(c_i) = \sum_{i \in I} \tau_M(\varphi_i \otimes c_i) = \tau_M \left( \sum_{i \in I} \varphi_i \otimes c_i \right).$$

Thus $\tau_M$ is surjective. □

### 3.2 Toric Varieties

For the remainder of this section we will consider the rings $S := F[A, B, C, X, Y, Z]/\overline{I}$, where again $\overline{I}$ is the ideal generated by the $2 \times 2$ minors of the matrix $\begin{pmatrix} A & B & X & Y \\ C & A & Z & X \end{pmatrix}$, and $R := S \otimes_{k[x_1, \ldots, x_s]}$. Note that $S$ and $R$ are naturally finitely generated graded $F$-algebras. Let $m_S$ and $m_R$ denote their irrelevant ideals, and note $S_\infty$ and $R_\infty$ are the completions of $S$ and $R$ at these ideals.

The goal of this subsection and the next one is to prove the following “de-completed” version of Theorem 3.3. In Section 3.4 we will show that this implies Theorem 3.3, and hence Theorem 3.1.

**Theorem 3.5.** If $M$ is a finitely-generated module over $R$ satisfying:

1. $M$ is maximal Cohen–Macaulay over $R$
2. We have $M^* \cong M$.
3. $g.\text{rank}_R M = 1$. 

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then $\dim_F M/m_R = 2^k$. Moreover, the trace map $\tau_M : M \otimes_R M \to \omega_R$ is surjective.

As outlined above, we will prove this theorem by computing $\Cl(R)$. The key insight that allows us to perform this computation is that $\mathcal{S}$, and hence $\mathcal{R}$, is the coordinate ring of an affine toric variety.

In this section, we review the basic theory of toric varieties and show that $\mathcal{S}$ and $\mathcal{R}$ indeed correspond to toric varieties. We shall primarily follow the presentation of toric varieties from [CLS11]. Unfortunately [CLS11] works exclusively with toric varieties over $\mathbb{C}$, whereas we are working in positive characteristic. All of the results we will rely on work over arbitrary base field, usually with identical proofs, so we will freely cite the results of [CLS11] as if they were stated over arbitrary fields. We refer the reader to [MS05] and [Dan78] for a discussion of toric varieties over arbitrary fields.

We recall the following definitions. For any integer $d \geq 1$, let $T_d = \mathbb{G}_m^d = (\mathbb{F}^\times)^d$, thought of as a group variety. Define the two lattices

$$M := \text{Hom}(T_d, \mathbb{G}_m) \quad \quad N := \text{Hom}(\mathbb{G}_m, T_d),$$

called the character lattice and the lattice of one-parameter subgroups, respectively. Note that $M \cong N \cong \mathbb{Z}^d$. We shall write $M$ and $N$ additively. For $m \in M$ and $u \in N$, we will write $\chi^m : T_d \to \mathbb{G}_m$ and $\lambda^u : \mathbb{G}_m \to T_d$ to denote the corresponding morphisms.

First note that there is a perfect pairing $\langle , \rangle : M \times N \to \mathbb{Z}$ given by $\langle m,u \rangle = \chi^m(\lambda^u(t))$. We shall write $M_R$ and $N_R$ for $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$, which are each $d$-dimensional real vector spaces. We will extend the pairing $\langle , \rangle$ to a perfect pairing $\langle , \rangle : M_R \times N_R \to \mathbb{R}$.

For the rest of this section, we will (arbitrarily) fix a choice of basis $e_1, \ldots, e_d$ for $M$, and so identify $M$ with $\mathbb{Z}^d$. We will also identify $N$ with $\mathbb{Z}^d$ via the dual basis to $e_1, \ldots, e_d$. Under these identifications, $\langle , \rangle$ is simply the usual (Euclidean) inner product on $\mathbb{Z}^d$.

We can now define:

**Definition 3.6.** An (affine) toric variety of dimension $d$ is a pair $(X, \iota)$, where $X$ is an affine variety $X/F$ of dimension $d$ and $\iota$ is an open embedding $\iota : T_d \hookrightarrow X$ such that the natural action of $T_d$ on itself extends to a group variety action of $T_d$ on $X$. We will usually write $X$ instead of the pair $(X, \iota)$.

For such an $X$, we define the semigroup of $X$ to be

$$S_X := \{ m \in M | \chi^m : T_d \to \mathbb{G}_m \text{ extends to a morphism } X \to \mathbb{A}^1 \} \subseteq M$$

For convenience, we will also say that a finitely generated $\mathbb{F}$-algebra $R$ (together with an inclusion $R \hookrightarrow \mathbb{F}[M]$) is toric if $\text{Spec} \ R$ is toric.

The primary significance of affine toric varieties is that they are classified by their semigroups. Specifically:
Proposition 3.7. If $X$ is an affine toric variety of dimension $d$, then $X = \text{Spec} \mathbb{F}[S_X]$, and the embedding $\iota: T_d \hookrightarrow X$ is induced by $\mathbb{F}[S_X] \hookrightarrow \mathbb{F}[M]$ (using the fact that $T_d = \text{Spec} \mathbb{F}[M]$). Moreover we have

1. The semigroup $S_X$ spans $M$ (that is, $\mathbb{Z}S_X$ has rank $d$).
2. If $S_X$ is saturated in $M$ (in the sense that $km \in S_X$ implies that $m \in S_X$ for all $k > 0$ and $m \in M$) then $X$ is a normal variety.

Conversely, if $S \subseteq M$ is a finitely generated semigroup spanning $M$ then the inclusion $\mathbb{F}[S] \hookrightarrow \mathbb{F}[M]$ gives $\text{Spec} \mathbb{F}[S]$ the structure of a $d$-dimensional affine toric variety.

Proof. cf. [CLS11] Proposition 1.1.14 and Theorems 1.1.17 and 1.3.5.

If $R$ is a toric $\mathbb{F}$-algebra, we will write $S_R$ to mean $S_{\text{Spec} R}$.

While it can be difficult to recognize toric varieties directly from Definition 3.6, the following Proposition makes it fairly easy to identify toric varieties in $\mathbb{A}^n$.

Proposition 3.8. Fix an integer $h \geq 1$ and let $\Phi: \mathbb{Z}^h \to M$ be any homomorphism with finite cokernel, and let $L = \ker \Phi$. Let $S \subseteq M$ be the semigroup generated by $\Phi(e_1), \ldots, \Phi(e_h) \in M$. Then we have an isomorphism $\mathbb{F}[z_1, \ldots, z_h]/I_L \cong \mathbb{F}[S]$ given by $z_i \mapsto \Phi(e_i)$, where

$$I_L := \left( z^\alpha - z^\beta \mid \alpha, \beta \in \mathbb{Z}^h_{\geq 0} \text{ such that } \alpha - \beta \in L \right) \subseteq \mathbb{F}[z_1, \ldots, z_h].$$

(Where, for any $\alpha = (\alpha_1, \ldots, \alpha_h) \in \mathbb{Z}^h_{\geq 0}$, we write $z^\alpha := z_1^{\alpha_1} \cdots z_h^{\alpha_h} \in \mathbb{F}[z_1, \ldots, z_h]$.)

Moreover $I_L$ can be explicitly computed as follows: Assume that $L = (\ell^1, \ldots, \ell^r)$ is a $\mathbb{Z}$-basis for $L$, with $\ell^i = (\ell^1_i, \ldots, \ell^r_i) \in \mathbb{Z}^h$. Write each $\ell^i$ as $\ell^i = \ell^i_+ - \ell^i_-$ where

$$\ell^i_+ = \left( \max\{\ell^1_i, 0\}, \ldots, \max\{\ell^r_i, 0\} \right) \in \mathbb{Z}_{\geq 0}^h$$

$$\ell^i_- = \left( \max\{-\ell^1_i, 0\}, \ldots, \max\{-\ell^r_i, 0\} \right) \in \mathbb{Z}_{\geq 0}^h.$$

Then if $I_L := \left( z^{\ell^i_+} - z^{\ell^i_-}, \ldots, z^{\ell^r_+} - z^{\ell^r_-} \right) \subseteq \mathbb{F}[z_1, \ldots, z_h]$, $I_L$ is the saturation of $I_L$ with respect to $z_1 \cdots z_h$, that is:

$$I_L = (I_L : (z_1 \cdots z_h)^\infty) := \left\{ f \in \mathbb{F}[z_1, \ldots, z_h] \mid (z_1 \cdots z_h)^m f \in I_L \text{ for some } m \geq 0 \right\}$$

$$= \left( z^\alpha - z^\beta \mid \alpha, \beta \in \mathbb{Z}^h_{\geq 0} \text{ such that } (z_1 \cdots z_h)^m (z^\alpha - z^\beta) \in I_L \text{ for some } m \geq 0 \right).$$

Conversely, if $I \subseteq \mathbb{F}[z_1, \ldots, z_h]$ is any prime ideal which can be written in the form $I = \left( z^{\alpha_i} - z^{\beta_i} \mid i \in \mathcal{A} \right)$ for a finite index set $\mathcal{A}$ and $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}^h$, then $I = I_L$ for some $L$ and so $\mathbb{F}[z_1, \ldots, z_h]/I$ can be given the structure of a toric $\mathbb{F}$-algebra.
Proof. This mostly follows from [CLS11, Propositions 1.1.8, 1.1.9 and 1.1.11]. The statement that \( I_L = (I_L : (z_1 \cdots z_h)^\infty) \) is [CLS11, Exercise 1.1.3] or [MS05, Lemma 7.6]

Now applying the above results to the \( \mathbb{F} \)-algebra \( \mathcal{S} \), we get:

**Proposition 3.9.** \( \mathcal{S} \) may be given the structure of a 3-dimensional toric \( \mathbb{F} \)-algebra, with semigroup

\[
\mathcal{S}_S := \{(a, b, c) \in \mathbb{Z}^3 | a, b, c \geq 0, 2a + 2b \geq c \} \subseteq M
\]

under some choice of basis \( e_1, e_2, e_3 \) for \( M \). Moreover:

1. \( \text{Spec} \mathcal{S} \) is the affine cone over a surface \( \mathcal{V} \subseteq \mathbb{P}^5 \) isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). The embedding \( \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathcal{V} \subseteq \mathbb{P}^5 \) corresponds to the very ample line bundle \( \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \).
2. \( \mathcal{S} \) is isomorphic to the ring \( \mathbb{F}[x, xz, xz^2, y, yz, yz^2] \subseteq \mathbb{F}[x, y, z] \).
3. \( \text{Spec} \mathcal{S} \) is a normal variety.
4. \( R \) is toric of dimension \( 3k + s \).

**Proof.** Take \( d = 3 \) in the above discussion, and fix an isomorphism \( M \cong \mathbb{Z}^3 \).

Write \( \mathcal{S} := \{(a, b, c) \in \mathbb{Z}^3 | a, b, c \geq 0, 2a + 2b \geq c \} \). We will first show that \( \mathcal{S} \cong \mathbb{F}[S] \). Note that \( \mathcal{S} \) is generated by the (transposes of) the columns of the matrix

\[
\Phi = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 2 & 0 & 1 & 2 & 0
\end{pmatrix},
\]

which in particular gives an isomorphism \( \mathbb{F}[S] \cong \mathbb{F}[x, xz, xz^2, y, yz, yz^2] \subseteq \mathbb{F}[x, y, z] \).

Let \( L = \ker \Phi \). By Proposition 3.8 it follows that \( \mathbb{F}[S] \cong \mathbb{F}[A, B, C, X, Y, Z]/I_L \) (where we have identified the ring \( \mathbb{F}[z_1, z_2, z_3, z_4, z_5, z_6] \) with \( \mathbb{F}[A, B, C, X, Y, Z] \) in the obvious way, in order to keep notation consistent).

But now note that \( L \) is a rank 3 lattice with basis \( L = (\ell^1, \ell^2, \ell^3) \) given by the vectors:

\[
\ell^1 := \begin{pmatrix}
1 \\
0 \\
-1 \\
-1 \\
0 \\
1
\end{pmatrix}, \quad \ell^2 := \begin{pmatrix}
1 \\
0 \\
-1 \\
1 \\
-1 \\
0
\end{pmatrix}, \quad \ell^3 := \begin{pmatrix}
1 \\
-1 \\
0 \\
1 \\
0 \\
-1
\end{pmatrix}.
\]

It follows that

\[
I_L = (AZ - CX, AX - CY, AX - BZ)
\]

from whence it is straightforward to compute that

\[
I_L = (I_L : (ABCXYZ)^\infty)
\]

\[
= (A^2 - BC, AZ - CX, AX - CY, AX - BZ, AY - BX, X^2 - YZ) = \mathcal{T}.
\]
Thus $S = \mathbb{F}[A, B, C, X, Y, Z]/\mathcal{T}$ is indeed toric with $S_S = S$, and we have

$$S \cong \mathbb{F}[S_S] \cong \mathbb{F}[x, xz, xz^2, y, yz, yz^2].$$

Moreover, this isomorphism sends the ideal $(A, B, C, X, Y, Z) \subseteq S$ to the ideal $(x, xz, xz^2, y, yz, yz^2)$, from which (2) easily follows.

Now the semigroup $S_S = \{(a, b, c) \in \mathbb{Z}^3 | a, b, c \geq 0, 2a + 2b \geq c \geq 0\}$ is clearly saturated in $M$, and so $\text{Spec } S$ is indeed normal, proving (3).

Now note that $\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ is indeed a very ample line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ and corresponds to the (injective) morphism $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5$ defined by

$$f([s : t], [u : v]) = [s^2u : stu : t^2u : s^2v : stv : t^2v].$$

It thus follows that the coordinate ring on the cone over the image of $f$ is isomorphic to

$$\mathbb{F}[s^2u, stu, t^2u, s^2v, stv, t^2v] \cong \mathbb{F}[x, xz, xz^2, y, yz, yz^2],$$

proving (1).

Lastly, recalling that $M = \mathbb{Z}^3$, let $M' = M = M^\otimes k \oplus Z^s = \mathbb{Z}^{3k+s}$ and define:

$$S' = (S)^\otimes k \otimes \mathbb{Z}_{\geq 0}^s$$

so that

$$\mathbb{F}[S'] = \mathbb{F}[S]^\otimes k \otimes \mathbb{F}[x]^\otimes s \cong S^\otimes k[x_1, \ldots, x_s] = R$$

so that $R$ is indeed toric of dimension $3k + s$, proving (4).

We will now restrict our attention to normal affine toric varieties. The advantage to doing this is that Proposition 3.7 has a refinement (see Proposition 3.11 below) that allows us to characterize normal toric varieties much more simply, using cones instead of semigroups.

We now make the following definitions:

**Definition 3.10.** A **convex rational polyhedral cone** in $N_\mathbb{R}$ is a set of the form:

$$\sigma = \text{Cone}(S) := \left\{ \sum_{\lambda \in S} x_\lambda \lambda \bigg| x_\lambda \geq 0 \text{ for all } \lambda \in S \right\} \subseteq N_{\mathbb{R}}$$

for some finite subset $S \subseteq N$.

A face of $\sigma$ is a subset $\tau \subseteq \sigma$ which can be written as $\tau = \sigma \cap H$ for some hyperplane $H \subseteq N_{\mathbb{R}}$ which does not intersect the interior of $\sigma$. We write $\tau \preceq \sigma$ to say that $\tau$ is a face of $\sigma$. It is clear that any face of $\sigma$ is also a convex rational polyhedral cone. We say that $\sigma$ is **strongly convex** if $\{0\}$ is a face of $\sigma$. 
We write $\mathbb{R}\sigma$ for the subspace of $N_\mathbb{R}$ spanned by $\sigma$, and we will let the dimension of $\sigma$ be $\dim \sigma := \dim_\mathbb{R} \mathbb{R}\sigma$.

We make analogous definitions for cones in $M_\mathbb{R}$.

For a convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$ (or similarly for $\sigma \subseteq M_\mathbb{R}$), we define its dual cone to be:

$$\sigma^\vee := \{ m \in M_\mathbb{R} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \}.$$  

It is easy to see that $\sigma^\vee$ is also convex rational polyhedral cone. If $\sigma$ is strongly convex and $\dim \sigma = d$ then the same is true of $\sigma^\vee$. Moreover, for any $\sigma$ we have $\sigma^\vee \vee = \sigma$.

We now have the following:

**Proposition 3.11.** If $X$ is a normal affine toric variety of dimension $d$, then there is a (uniquely determined) strongly convex rational polyhedral cone $\sigma_X \subseteq N_\mathbb{R}$ for which $\sigma_X \cap M = S_X$ and $\sigma_X \cap N = \{ u \in N \mid \lambda^u : \mathbb{G}_m \to T_d \text{ extends to a morphism } \mathbb{A}^1 \to X \}$

$$= \left\{ u \in N \mid \lim_{t \to 0} \lambda^u(t) \text{ exists in } X \right\}.$$  

We call $\sigma_X$ the cone associated to $X$. Again, if $R$ is a toric $\mathbb{F}$-algebra, then we write $\sigma_R$ for $\sigma_{\text{Spec } R}$.

**Proof.** This follows from [CLS11] Theorem 1.3.5 (for $\sigma_X \cap M$) and Proposition 3.2.2 (for $\sigma_X \cap N$). Note that it is clear from our definitions that a convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$ is uniquely determined by $\sigma \cap N$. \hfill \square

**Remark.** Based on the statement of Proposition 3.11, it would seem more natural to simply define the cone associated to $X'$ to be $\sigma_X^\vee$, and not mention the lattice $N$ at all. The primary reason for making this choice in the literature is to simplify the description of non-affine toric varieties, which is not relevant to our applications. Nevertheless we shall use the convention established in Proposition 3.11 to keep our treatment compatible with existing literature, and specifically to avoid having to reformulate Theorem 3.13, below.

Rephrasing the statement of Proposition 3.9 in terms of cones, we get:

**Corollary 3.12.** We have $\sigma_S = \text{Cone}(e_1, e_2, e_3, 2e_1 + 2e_2 - e_3)$.

**Proof.** The description of $S_X$ in Proposition 3.9 immediately implies that

$$\sigma_X^\vee = \{ (x, y, z) \in \mathbb{R}^3 \mid x, y, x \geq 0, 2x + 2y + z \geq 0 \} = \text{Cone}(e_1, e_2, e_1 + 2e_3, e_2 + 2e_3).$$  

Thus we get

$$\sigma_X = \sigma_X^{\vee \vee} = \{ u \in N \mid \langle m, u \rangle \geq 0 \text{ for all } m \in \sigma_X^\vee \}$$  

$$= \{ u \in N \mid \langle m, e_1 \rangle, \langle m, e_2 \rangle, \langle m, e_1 + 2e_3 \rangle, \langle m, e_2 + 2e_3 \rangle \geq 0 \}$$  

$$= \{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, x + 2z \geq 0, y + 2z \geq 0 \}$$  

$$= \text{Cone}(e_1, e_2, e_1 + 2e_3, e_2 + 2e_3).$$

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3.3 Class Groups of Toric Varieties

The benefit of this entire discussion is that Weil divisors on toric varieties are much easier work with than they are for general varieties. In order to explain this, we first introduce a few more definitions.

For any variety $X$, we will let $\text{Div}(X)$ denote the group of Weil divisors of $X$. If $X = \text{Spec} \mathcal{O}$ is normal, affine and toric of dimension $d$, then the torus $T_d$ acts on $X$, and hence acts on $\text{Div}(X)$. We say that a divisor $D \in \text{Div}(X)$ is torus-invariant if it is preserved by this action. We will write $\text{Div}_{T_d}(X) \subseteq \text{Div}(X)$ for the group of torus invariant divisors.

Now consider the (strongly convex, rational polyhedral) cone $\sigma_R \subseteq \mathbb{N}_\mathbb{R}$. We will let $\sigma_R(1)$ denote the set of edges (1 dimensional faces) of $\sigma_R$. For any $\rho \in \sigma_R(1)$, note that $\rho \cap N$ is a semigroup isomorphic to $\mathbb{Z}_{\geq 0}$, and so there is a unique choice of generator $u_\rho \in \rho \cap N$ (called a minimal generator). By Proposition 3.11, the limit $\gamma_\rho := \lim_{t \to 0} \lambda^{u_\rho}(t) \in X$ exists. Thus we may consider its orbit closure $D_\rho := T_d \cdot \gamma_\rho \subseteq X$.

The following theorem allows us to characterize $\text{Cl}(R)$, and $[\omega_R] \in \text{Cl}(R)$, entirely in terms of the set $\sigma_R(1)$.

**Theorem 3.13.** Let $X = \text{Spec} \mathcal{O}$ be a normal affine toric variety, with cone $\sigma_R \subseteq \mathbb{N}_\mathbb{R}$. We have the following:

1. For any $\rho \in \sigma_R(1)$, $D_\rho \subseteq X$ is a torus-invariant prime divisor. Moreover, $\text{Div}_{T_d}(X) = \bigoplus_{\rho \in \sigma_R(1)} \mathbb{Z}D_\rho$.

2. Any divisor $D \in \text{Div}(X)$ is rationally equivalent to a torus-invariant divisor.

3. For any $m \in M$, the rational function $\chi^m \in K(X)$ has divisor $\text{div}(\chi^m) = \sum_{\rho \in \sigma_R(1)} \langle m, u_\rho \rangle D_\rho$.

4. For any torus-invariant divisor $D$,

   $$O(D) := \{ f \in K(X) \mid \text{div}(f) + D \geq 0 \} = \bigoplus_{\chi^m \in O(D)} \mathbb{F} \chi^m = \bigoplus_{\text{div}(\chi^m) + D \geq 0} \mathbb{F} \chi^m \subseteq K(X)$$

5. There is an exact sequence

   $$M \rightarrow \text{Div}_{T_d}(X) \rightarrow \text{Cl}(R) \rightarrow 0$$

   where the first map is $m \mapsto \text{div}(\chi^m)$ and the second map is $D \mapsto O(D)$.

6. $R$ is Cohen–Macaulay and we have $\omega_R \cong \mathcal{O} \left( - \sum_{\rho \in \sigma_R(1)} D_\rho \right)$

**Proof.** By the orbit cone correspondence ([CLS11] Theorem 3.2.6), it follows that each $D_\rho$ is a torus-invariant prime divisor, and moreover that these are the only torus-invariant prime divisors. The rest of (1) follows easily from this (cf. [CLS11] Exercise 4.1.1).
Proposition 3.14. Let \( e_0 = 2e_1 + 2e_2 - e_3 \), so that \( \sigma_X = \text{Cone}(e_0, e_1, e_2, e_3) \). For each \( i \), let \( \rho_i = \mathbb{R}_{\geq 0}e_i \) and \( D_i = D_{\rho_i} \), so that \( u_{\rho_i} = e_i \) and \( \sigma_X(1) = \{ \rho_0, \rho_1, \rho_2, \rho_3 \} \). Then:

1. We have an isomorphism \( \text{Cl}(S) \cong \mathbb{Z} \) given by \( k \mapsto \mathcal{O}(kD_0) \).
2. \( \omega_S \cong \mathcal{O}(2D_0) \).
3. If \( \mathcal{M} \) is a generic rank 1 reflexive, self-dual module over \( S \), then \( \mathcal{M} \cong \mathcal{O}(D_0) \).
4. Identifying \( S \) with \( \mathbb{F}[x, xz, xz^2, y, yz, yz^2] \subseteq \mathbb{F}[x, y, z] \) as in Proposition 3.9 we get

\[
\mathcal{O}(D_0) \cong S \cap xz\mathbb{F}[x, y, z] = (xz, xz^2) \subseteq S
\]

\[
\omega_S = \mathcal{O}(2D_0) \cong S \cap x\mathbb{F}[x, y, z] = (x, xz, xz^2) \subseteq S,
\]

so in particular, \( \dim_{\mathbb{F}} \mathcal{O}(D_0)/m_S = 2 \).
5. There is a surjection \( \mathcal{O}(D_0) \otimes S \mathcal{O}(D_0) \to \omega_S \).

Proof. Write \( x = \chi^{e_1}, y = \chi^{e_2} \) and \( z = \chi^{e_3} \), so that \( \mathbb{F}[M] = \mathbb{F}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}] \). By Theorem 3.13(3) we get that

\[
\text{div}(x) = 2D_0 + D_1, \quad \text{div}(y) = 2D_0 + D_2, \quad \text{div}(z) = D_3 - D_0.
\]

It follows that \( D_1 \sim -2D_0, D_2 \sim -2D_0 \) and \( D_3 \sim D_0 \), and so from Theorem 3.13(5) we get that \( \text{Cl}(S) \) is generated by \( \mathcal{O}(D_0) \). Moreover, the exactness in Theorem 3.13(5) gives that the above relations are the only ones between the \( D_i \)'s, and so \( \mathcal{O}(D_0) \) is non-torsion in \( \text{Cl}(S) \), indeed giving the isomorphism \( \text{Cl}(S) \cong \mathbb{Z} \). (Alternatively, Theorem 3.13(5) implies that the \( \mathbb{Z} \)-rank of \( \text{Cl}(S) \) is at least \( \text{rank} \text{div}_{\mathcal{X}}(\mathcal{X}) - \text{rank} M = 4 - 3 = 1 \).) This proves (1).

For (2), we simply use Theorem 3.13(6):

\[
\omega_S \cong \mathcal{O}(-D_0 - D_1 - D_2 - D_3) \cong \mathcal{O}(-D_0 - (-2D_0) - (-2D_0) - D_0) = \mathcal{O}(2D_0).
\]

Now by (1), any generic rank 1 reflexive \( S \)-module is in the form \( \mathcal{O}(kD_0) \) for some \( k \in \mathbb{Z} \), and by (2) \( \mathcal{O}(kD_0)^* \cong \mathcal{O}((2 - k)D_0) \). Thus if \( \mathcal{O}(kD_0) \) is self-dual then \( k = 1 \), giving (3).

By the above computations, we get that

\[
\text{div}(x^ay^bz^c) = (2a + 2b - c)D_0 + 2aD_1 + 2bD_2 + cD_3
\]

for any \( (a, b, c) \in \mathbb{Z}^3 \). Now note that \( \mathcal{O}(D_0) \cong \mathcal{O}(-D_1 - D_3) \) and \( \mathcal{O}(2D_0) \cong \mathcal{O}(-D_1) \) which are
both ideals of $S$. But now by Theorem 3.13(4) as ideals of $S$ we have
\[
\mathcal{O}(D_0) \cong \mathcal{O}(-D_1 - D_3) = (x^ay^bz^c \mid 2a + 2b - c \geq 0, 2a \geq 1, 2b \geq 0, c \geq 1)
\]
\[
= (x^ay^bz^c \mid 2a + 2b - c \geq 0, a \geq 1, b \geq 0, c \geq 1)
\]
\[
= (x^ay^bz^c \mid x^ay^bz^c \in S, xz \mid x^ay^bz^c)
\]
\[
= S \cap xzF[x, y, z] = (xz, xz^2)
\]
\[
\mathcal{O}(2D_0) \cong \mathcal{O}(-D_1) = (x^ay^bz^c \mid 2a + 2b - c \geq 0, 2a \geq 1, 2b \geq 0, c \geq 0)
\]
\[
= (x^ay^bz^c \mid 2a + 2b - c \geq 0, a \geq 1, b \geq 0, c \geq 0)
\]
\[
= (x^ay^bz^c \mid x^ay^bz^c \in S, x \mid x^ay^bz^c)
\]
\[
= S \cap xF[x, y, z] = (x, xz, xz^2),
\]
proving (4).

Now identify $\mathcal{O}(D_0)$ with $(xz, xz^2) \subseteq S$ and $\omega_S$ with $(x, xz, xz^2) \subseteq S$. Notice that
\[
\mathcal{O}(D_0)\mathcal{O}(D_0) = (xz, xz^2)(xz, xz^2) = (x^2z^2, x^2z^3, x^2z^4) = xz(x, xz, xz^2) = xz^2\omega_S.
\]

Thus we can define a surjection $f : \mathcal{O}(D_0) \otimes_S \mathcal{O}(D_0) \to \omega_S$ by $f(\alpha \otimes \beta) = \frac{1}{xz^2}\alpha\beta$, proving (5).

We can now compute $\text{Cl}(R)$ and $\omega_R$, by using the following lemma:

**Lemma 3.15.** For any normal, affine toric varieties $X$ and $Y$ the natural map $\text{Cl}(X) \oplus \text{Cl}(Y) \to \text{Cl}(X \times Y)$ given by $([A], [B]) \mapsto [A \boxtimes B]$ is an isomorphism which sends $([\omega_X], [\omega_Y])$ to $\omega_{X \times Y}$.

**Proof.** By [CLS11] Proposition 3.1.14, $X \times Y$ is a toric variety with cone $\sigma_{X \times Y} = \sigma_X \times \sigma_Y$. It follows that $\sigma_{X \times Y}(1) = \sigma_X(1) \cup \sigma_Y(1)$. The claim now follows immediately from Theorem 3.13.

Thus we have:

**Corollary 3.16.** The map $\varphi : \text{Cl}(S)^k \to \text{Cl}(R)$ given by
\[
(\ldots, [A_k]) \mapsto [(A_1 \boxtimes A_2 \boxtimes \ldots \boxtimes A_k)[x_1, \ldots, x_s]]
\]
is an isomorphism which sends $([\omega_S], \ldots, [\omega_S])$ to $[\omega_R]$.

Consequently there is a unique self-dual generic rank 1 reflexive module $M$ over $R$, which is the image of $([O(D_0)], \ldots, [O(D_0)])$. We have that $\dim_S M/m_R = 2^k$ and there is a surjection $M \otimes_R M \to \omega_R$. 

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Proof. The isomorphism follows immediately from Corollary 3.15 (noting that \(A^1\) is a toric variety with \(\text{Cl}(A^1) = 0\) and \(\omega_{A^1} = A^1\)).

Now for any self-dual generic rank 1 reflexive module \(\mathcal{M}\) over \(\mathcal{R}\), it follows that \([\mathcal{M}] = \varphi([A_1], \ldots, [A_a])\) where each \(A_i\) is self-dual. Proposition 3.14 implies that each \(A_i\) is isomorphic to \(O(D_0)\), as claimed.

For this \(\mathcal{M}\) we indeed have
\[
\mathcal{M}/m_{\mathcal{R}} = \mathcal{O}(D_0)^{\otimes k} \left[ x_1, \ldots, x_s \right] \equiv \left( \mathcal{O}(D_0) \right)^{\otimes k} \left[ m_{\mathcal{S}} \right] = (F^2)^{\otimes k} = F^{2k}.
\]

Also, the surjection \(\mathcal{O}(D_0) \otimes_\mathcal{S} \mathcal{O}(D_0) \to \omega_\mathcal{S}\) from Proposition 3.14 indeed gives a surjection
\[
\mathcal{M} \otimes_\mathcal{R} \mathcal{M} = \left( \mathcal{O}(D_0)^{\otimes k} \left[ x_1, \ldots, x_s \right] \right) \otimes_\mathcal{R} \left( \mathcal{O}(D_0)^{\otimes k} \left[ x_1, \ldots, x_s \right] \right)
\equiv (\mathcal{O}(D_0) \otimes \mathcal{O}(D_0))^{\otimes k} \left[ x_1, \ldots, x_s \right] \to \omega_\mathcal{S}^{\otimes k} \left[ x_1, \ldots, x_s \right] \equiv \omega_\mathcal{R}.
\]

Which completes the proof of Theorem 3.5.

Remark. In our proof of Theorem 3.5, we never actually used the first condition, namely that \(\mathcal{M}\) was maximal Cohen–Macaulay over \(\mathcal{R}\). We only used the (strictly weaker) assumption that \(\mathcal{M}\) was reflexive, which, combined with the fact that \(\mathcal{M}\) was self-dual, was enough to uniquely determine the structure of \(\mathcal{M}\).

In most situations, the modules \(M_\infty\) produced by the patching method will be maximal Cohen–Macaulay, but it is possible that they might fail to be self-dual (e.g. if they arise from the cohomology of a non self-dual local system).

In this situation it is possible to formulate a weaker version of 3.1, where one drops the self-duality assumption. Specifically one can show (in the notation of Proposition 3.14) that the only Cohen–Macaulay generic rank one modules over the ring \(\mathcal{S}\) are the 5 modules:

\[
\mathcal{O}(-D_0) = (xz, xz^2, yz, yz^2) \\
\mathcal{O} = \mathcal{S} \\
\mathcal{O}(D_0) = (xz, xz^2) \\
\mathcal{O}(2D_0) = (x, xz, xz^2) \\
\mathcal{O}(3D_0) = (x^2, x^2z, x^2z^2, x^2z^3).
\]

This can be done quite simply by first localizing at \(m_{\mathcal{S}}\), and noting that any regular sequence for \(\mathcal{S}\), such as \((x, yz^2, y - xz^2)\), must also be a regular sequence for \(\mathcal{M}_m\) over \(\mathcal{S}\) where \(\mathcal{M}\) is any maximal Cohen–Macaulay module over \(\mathcal{S}\). Thus if \(g, \text{rank}_S \mathcal{M} = 1\) we must have
\[
\dim_F \mathcal{M}/m_{\mathcal{S}} \mathcal{M} = \dim_F \mathcal{M}_m \mathcal{S}/m_{\mathcal{S}} \mathcal{M}_m \leq \dim_F \mathcal{M}/(x, yz^2, y - xz^2) = \dim_F \mathcal{S}/(x, yz^2, y - xz^2) = 4,
\]
26
reducing the problem to simply checking that the above 5 modules are indeed all maximal Cohen–Macaulay.

This unfortunately does not allow us to uniquely deduce the structure of $M$ and hence of $M_{\infty}$, but it does give us the bound $\dim_{F} M_{\infty}/m_{R_{\infty}} \leq 4^k$, and could potentially lead to more refined information about $M_{\infty}$, which may be of independent interest.

### 3.4 Class groups of completed rings

The goal of this section is to prove that Theorem 3.5 implies Theorem 3.3. We shall do this by proving that the natural map $\text{Cl}(R) \to \text{Cl}(R_{\infty})$ given by $[M] \mapsto [M \otimes_{R} R_{\infty}]$ is an isomorphism.

First note that the Theorem 3.3 will indeed follow from this. Assume that $M_{\infty}$ is an $R_{\infty}$-module satisfying the conditions of Theorem 3.3. Then in particular it corresponds to an element of $\text{Cl}(R_{\infty})$, and so there is some reflexive generic rank 1 $R$-module $M$ with $M_{\infty} \sim M \otimes_{R} R_{\infty} = \varprojlim M/m_{R}^{n} M$. We claim that $M$ satisfies conditions (1) and (2) of Theorem 3.5 (by assumption, it satisfies condition (3)).

Showing that $M$ is self-dual is equivalent to showing that $2[M] = [\omega_{R}]$ in $\text{Cl}(R)$, which follows from the fact that $2[M \otimes_{R} R_{\infty}] = 2[M_{\infty}] = [\omega_{R_{\infty}}]$ in $\text{Cl}(R_{\infty})$ and the fact that $\omega_{R_{\infty}} \sim \omega_{R} \otimes_{R} R_{\infty}$ (cf [Eis95, Corollaries 21.17 and 21.18]).

We now observe that $M$ is maximal Cohen–Macaulay over $R$. By Theorem 3.2, $(C,Y,B - Z)$ is a regular sequence for $S$ consisting of homogeneous elements. It follows that this is also a regular sequence for $R$, and so $R$ also has a regular sequence $(z_{1}, \ldots, z_{3k+s})$ consisting entirely of homogeneous elements. Now it follows that this regular sequence is also regular for $R_{\infty}$, and hence for $M_{\infty}$. But now as the $z_{i}$'s are all homogeneous it follows that $M/(z_{1}, \ldots, z_{i}) \to M_{\infty}/(z_{1}, \ldots, z_{i})$ for all $i$ and so $(z_{1}, \ldots, z_{3k+s})$ is also a regular sequence for $M$. Hence $M$ is maximal Cohen–Macaulay over $R$.

Hence $M$ satisfies the conditions of Theorem 3.5, so we get that $\dim_{F} M/m_{R} M = 2^k$ and $\tau_{M} : M \otimes_{R} M \to \omega_{R}$ is surjective.

Now as $M_{\infty}/m_{R_{\infty}} M_{\infty} \cong M/m_{R} M$, we indeed get $\dim_{F} M_{\infty}/m_{R_{\infty}} M_{\infty} = 2^k$.

To show that $\tau_{M_{\infty}}$ is surjective, note that

$$(M \otimes_{R} M) \otimes_{R} R_{\infty} \cong (M \otimes_{R} R_{\infty}) \otimes_{R_{\infty}} (M \otimes_{R} R_{\infty}) \cong M_{\infty} \otimes_{R_{\infty}} M_{\infty}$$

so as $\omega_{R}$ is a quotient of $M \otimes_{R} M$ it follows that $\omega_{R_{\infty}} \cong \omega_{R} \otimes_{R} R_{\infty}$ is a quotient of $M_{\infty} \otimes_{R_{\infty}} M_{\infty}$ and so $\tau_{M_{\infty}}$ is indeed surjective by Lemma 3.4, which completes the proof of Theorem 3.3.

Unfortunately, it is not true in general that if $R$ is a graded $F$-algebra and $\hat{R}$ is the completion at

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7 Strictly speaking it is not necessary to show this, as condition (1) was never used in the proof of Theorem 3.5, but we will still show it for the sake of completeness.
the irrelevant ideal then the map \( \mathrm{Cl}(R) \to \mathrm{Cl}(\hat{R}) \) is an isomorphism. However Danilov [Dan68] has shown that this is true in certain cases:

**Theorem 3.17** (Danilov). Let \( V \) be a smooth projective variety with a very ample line bundle \( \mathcal{L} \) giving an injection \( V \hookrightarrow \mathbb{P}^N \). Let \( \text{Spec} \, S \subseteq \mathbb{A}^{N+1} \) be the affine cone on \( V \), so that \( S \) is a graded \( \mathbb{F} \)-algebra, and let \( \hat{S} \) be the completion of \( R \) at the irrelevant ideal. Then:

1. The natural map \( \mathrm{Cl}(S) \to \mathrm{Cl}(\hat{S}) \) is an isomorphism if and only if \( H^1(V, \mathcal{L}^{\otimes i}) = 0 \) for all \( i \geq 1 \).
2. For \( s > 0 \), the natural map \( \mathrm{Cl}(S) \to \mathrm{Cl}(\hat{S}[[x_1, \ldots, x_s]]) \) is an isomorphism if and only if \( H^1(V, \mathcal{L}^{\otimes i}) = 0 \) for all \( i \geq 0 \).

We now make the following observation:

**Lemma 3.18.** There exists a smooth projective variety \( V \) and an ample line bundle \( \mathcal{L} \) on \( V \) such that \( \text{Spec} \, S \) is the affine cone over \( V \), under the projective embedding induced by \( \mathcal{L} \). We have \( H^d(V, \mathcal{L}^{\otimes i}) = 0 \) for all \( d \geq 1 \) and \( i \geq 0 \).

**Proof.** This is largely a restatement of Proposition 3.9(1). Specifically we have \( V = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \). To prove the vanishing of cohomology, we simply note that \( H^d(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = 0 \) for all \( d \geq 1 \) and \( i \geq 0 \), and so

\[
H^d(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2 \otimes \mathcal{O}_{\mathbb{P}^1}(i))) = \bigoplus_{e=0}^{d} H^e(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2i)) \otimes H^{d-e}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) = 0
\]

for any \( d \geq 1 \) and \( i \geq 0 \). \( \square \)

**Remark.** By [Eis95, Exercise 18.16], the conclusion of Lemma 3.18 about the vanishing of the cohomology groups \( H^d(V, \mathcal{L}^{\otimes i}) \) holds whenever \( S \) is Cohen–Macaulay and \( d < \dim V \). This means the results of this section will be applicable in most cases where the ring \( R_\infty \) is Cohen–Macaulay, and so this does not impose a fundamental limitation on our method.

It follows that \( \mathrm{Cl}(S) \cong \mathrm{Cl}(\hat{S}) \). In fact (as the natural map \( \mathrm{Cl}(S) \to \mathrm{Cl}(S[[x_1, \ldots, x_s]]) \) is an isomorphism) it follows that the natural map \( \mathrm{Cl}(S[[x_1, \ldots, x_s]]) \to \mathrm{Cl}(\hat{S}[[x_1, \ldots, x_s]]) \) is an isomorphism, and so Theorem 3.3 follows in the case when \( k = 1 \).

When \( k > 1 \) however, we cannot directly appeal to Theorem 3.17, as \( \text{Spec} \, R \) is no longer the cone over a smooth projective variety, and in fact \( \text{Spec} \, R \) does not have isolated singularities. Fortunately it is fairly straightforward to adapt the method of [Dan68] to our situation. Specifically, we will prove the following (which obviously applies to the ring \( R \)):

**Proposition 3.19.** Let \( V_1, \ldots, V_k \) be a collection of smooth projective varieties of dimension at least 1, and for each \( j \), let \( \mathcal{L}_j \) be a very ample line bundle on \( V_j \) giving an injection \( V_j \hookrightarrow \mathbb{P}^{N_j} \). Let \( \text{Spec} \, S_j \subseteq \mathbb{A}^{N_j+1} \) be the affine cone on \( V_j \). Let

\[
R = \bigotimes_{j=1}^{k} S_j \quad \left[x_1, \ldots, x_s\right]
\]
(for some \( s \geq 0 \)), so that \( R \) is a graded \( \mathbb{F} \)-algebra. Let \( \hat{R} \) be the completion of \( R \) at the irrelevant ideal.

If we have that \( H^d(V_j, \mathcal{L}^{\otimes i}_j) = 0 \) for all \( d = 1, 2, j = 1, \ldots, k \) and \( i \geq 0 \), then the natural map \( \text{Cl}(R) \to \text{Cl}(\hat{R}) \) is an isomorphism.

**Proof.** For simplicity, we first show we may reduce to the case \( s = 0 \). If \( s \geq 2 \), then we may simply let \( V_{k+1} = \mathbb{P}^{s-1}, \mathcal{L} = \mathcal{O}_{\mathbb{P}^{s-1}}(1) \) at note that we still have the cohomology condition \( H^d(V_{k+1}, \mathcal{L}^{\otimes i}_{k+1}) = H^d(\mathbb{P}^{s-1}, \mathcal{O}_{\mathbb{P}^{s-1}}(i)) = 0 \) for all \( d \geq 1 \) and \( i \geq 0 \). So the \( s \geq 2 \) case follows from the \( s = 0 \) case.

The \( s = 1 \) case now follows from the \( s = 0 \) and \( s \geq 2 \) cases by letting \( R_0 = \bigotimes_{j=1}^k S_j \) and considering the commutative diagram:

\[
\begin{array}{ccc}
\text{Cl}(R_0) & \longrightarrow & \text{Cl}(R_0[x_1]) \\
\downarrow & & \downarrow \\
\text{Cl}(\hat{R}_0) & \longrightarrow & \text{Cl}(\hat{R}_0[[x_1]])
\end{array}
\]

and noting that the maps on the top row are isomorphisms by standard properties of the class groups of varieties, and the maps on the bottom row are injective (since if \( M \) is a reflexive \( \hat{R}_0 \) module and \( M[[x_1]] = M \otimes_{\hat{R}_0} \hat{R}_0[[x_1]] \) is a free \( \hat{R}_0[[x_1]] \)-module, then \( M/m_{\hat{R}_0} \cong M[[x_1]]/m_{\hat{R}_0}[[x_1]] \cong \mathbb{F} \), and so \( M \) is a cyclic, and thus free \( \hat{R} \) module). So from now on, we shall assume \( s = 0 \).

We first introduce some notation.

For each \( j \), let \( Y_j := \text{Spec} S_j \). Let \( X := \coprod_{j=1}^k V_j \) and \( Y := \coprod_{j=1}^k Y_j = \text{Spec} R \). Also let \( Z_j := \coprod_{j' \neq j} Y_{j'} \subseteq Y \) and \( Z := Z_1 \cup Z_2 \cdots \cup Z_k \subseteq Y \). Note that each \( Z_j \) is irreducible subscheme of \( Y \) of codimension at least 2.

Write \( Z_j = \text{Spec} R/I_j \) and \( Z = \text{Spec} R/I \). Note that \( I_j = m_j R \), where \( m_j \) is the irrelevant ideal of \( S_j \), and \( I = I_1 I_2 \cdots I_k \). In particular, \( I_j \) and \( I \) are homogeneous ideals of \( R \). Now let \( \hat{Y} = \text{Spec} \hat{R}, \hat{I}_j = I_j \hat{R}, \hat{I} = I \hat{R}, \hat{Z}_j = \text{Spec} \hat{R}/\hat{I}_j \) and \( \hat{Z} = \text{Spec} \hat{R}/\hat{I} \). Note that the \( \hat{Z}_j \)'s are still irreducible, and we have \( \hat{Z} = \hat{Z}_1 \cup \hat{Z}_2 \cup \cdots \cup \hat{Z}_k \).

Now let \( C = \text{Proj} \left( \bigoplus_{n=0}^\infty I^n \right) \) and \( \hat{C} = \text{Proj} \left( \bigoplus_{n=0}^\infty \hat{I}^n \right) \) be the blowups of \( Y \) and \( \hat{Y} \) along \( Z \) and \( \hat{Z} \) and let \( p : C \to Y \) and \( \bar{p} : \hat{C} \to \hat{Y} \) be the projection maps. Let \( E_j = p^{-1}(Z_j), \tilde{E}_j = \bar{p}^{-1}(\hat{Z}_j), E = p^{-1}(Z) \) and \( \tilde{E} = \bar{p}^{-1}(\hat{Z}) \). Note that the \( E_j \)'s and \( \tilde{E}_j \)'s are irreducible and we have \( E = E_1 \cup E_2 \cup \cdots \cup E_k \) and \( \tilde{E} = \tilde{E}_1 \cup \tilde{E}_2 \cup \cdots \cup \tilde{E}_k \).
Let \( m_R \subseteq R \) denote the irrelevant ideal and let \( \hat{m}_R = m_R\hat{R} \subseteq R \) be its completion. Notice that we have natural isomorphisms \( p^{-1}(\{m_R\}) = \hat{E}_1 \cap E_2 \cap \cdots \cap E_k \cong X \) and \( \tilde{p}^{-1}(\{\hat{m}_R\}) = \hat{E}_1 \cap \hat{E}_2 \cap \cdots \cap \hat{E}_k \cong X \). Identify \( X \) with its images in both \( C \) and \( \hat{C} \). We will let \( \hat{C} \) and \( \tilde{C} \) denote the formal completions of \( C \) and \( \hat{C} \) along the subvariety \( X \).

Lastly, we define a rank \( k \) vector bundle \( \xi \) on \( X \) as follows. For each \( j \), let \( \pi_j X \to \mathcal{V}_j \) be the projection map, so that \( \pi_j^* \mathcal{L}_j = \mathcal{O}_{\mathcal{V}_1} \boxtimes \cdots \boxtimes \mathcal{L}_j \boxtimes \cdots \boxtimes \mathcal{O}_{\mathcal{V}_k} \) is a line bundle on \( X \). We will let \( \xi = \pi_1^* \mathcal{L}_1 \oplus \pi_2^* \mathcal{L}_2 \oplus \cdots \oplus \pi_k^* \mathcal{L}_k \).

We first observe the following:

**Lemma 3.20.** There is an isomorphism \( C \cong V(\xi) \), where \( V(\xi) \) is the total space of the vector bundle \( \xi \) over \( X \). This isomorphism is compatible with the inclusions \( X \hookrightarrow C \) and \( X \hookrightarrow V(\xi) \).

Moreover we have isomorphisms of formal schemes \( \hat{C} \cong \hat{V}(\xi) \cong \tilde{C} \), where \( \hat{V}(\xi) \) is the completion of \( V(\xi) \) along the zero section \( X \hookrightarrow V(\xi) \). These isomorphisms are again compatible with the natural inclusions of \( X \).

**Proof.** Letting \( V(\mathcal{L}_j) \) be the total space of \( \mathcal{L}_j \) over \( \mathcal{V}_j \) we see that

\[
V(\xi) = V(\pi_1^* \mathcal{L}_1 \oplus \pi_2^* \mathcal{L}_2 \oplus \cdots \oplus \pi_k^* \mathcal{L}_k) \cong V(\pi_1^* \mathcal{L}_1) \times_X V(\pi_2^* \mathcal{L}_2) \times_X \cdots \times_X V(\pi_k^* \mathcal{L}_k) \\
\cong V(\mathcal{L}_1) \times V(\mathcal{L}_2) \times \cdots \times V(\mathcal{L}_k).
\]

Now as in [Dan68, Lemma 1(3)], each \( V(\mathcal{L}_j) \) is the blowup of \( \text{Spec} \, S_j \) at the point \( m_j \). Now using this and the fact that \( I = I_1 I_2 \cdots I_k = m_1 \otimes m_2 \otimes \cdots \otimes m_k \) we indeed get

\[
V(\xi) \cong \prod_{j=1}^k V(\mathcal{L}_j) \cong \prod_{j=1}^k \text{Proj} \left( \bigoplus_{n=0}^{\infty} m_j^n \right) \cong \text{Proj} \left( \bigoplus_{n=0}^{\infty} (m_1^n \otimes m_2^n \otimes \cdots \otimes m_k^n) \right) = \text{Proj} \left( \bigoplus_{n=0}^{\infty} I^n \right) = C,
\]

where we used the fact that \( \text{Proj} \left( \bigoplus_{n=0}^{\infty} A_n \right) \times \text{Proj} \left( \bigoplus_{n=0}^{\infty} B_n \right) \cong \text{Proj} \left( \bigoplus_{n=0}^{\infty} A_n \otimes B_n \right) \) for finitely generated graded \( R \)-algebras \( \bigoplus_{n=0}^{\infty} A_n \) and \( \bigoplus_{n=0}^{\infty} B_n \). (See for instance, [Vak17, Exercise 9.6.D])

It is easy to check that these isomorphisms are compatible with the embeddings \( X \hookrightarrow C, V(\xi) \). This automatically gives \( \hat{C} \cong \hat{V}(\xi) \).

Now notice that the subscheme \( X = p^{-1}(\{m_R\}) \subseteq C \) is cut out by the ideal sheaf \( \mathcal{I} := p^*(m_R) \) and similarly the subscheme \( \hat{X} = \tilde{p}^{-1}(\{\hat{m}_R\}) \subseteq \hat{C} \) is cut out by the ideal sheaf \( \tilde{\mathcal{I}} := \tilde{p}^*(\hat{m}_R) \). But now using the fact that \( \hat{m}_R^n/\hat{m}_R^{n+1} = m_R^n/m_R^{n+1} \) for all \( a > b \), as in [Dan68, Section 4] we get that

\[
\hat{C} = \lim_{n} \left( X, \mathcal{O}_C/\mathcal{I}^{n+1} \right) = \lim_{n} \tilde{p}^{-1} \left( \text{Spec} \left( R/\hat{m}_R^{n+1} \right) \right) = \lim_{n} \text{Proj} \left( \bigoplus_{i=0}^{n} m_R^n/m_R^{n+1} \right) \\
= \lim_{n} \text{Proj} \left( \bigoplus_{i=0}^{n} \hat{m}_R^n/\hat{m}_R^{n+1} \right) = \lim_{n} p^{-1} \left( \text{Spec} \left( R/m_R^{n+1} \right) \right) = \lim_{n} \left( X, \mathcal{O}_C/\mathcal{I}^{n+1} \right) = \hat{C}.
\]
completing the proof of the lemma.

We next note the following analogue of [Dan68, Section 2]:

**Lemma 3.21.** Assume the setup of Proposition 3.19, but without the assumption about the vanishing of the cohomology groups. There is a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{Z}^k & \rightarrow & \text{Pic}(X) & \rightarrow & \text{Cl}(R) & \rightarrow & 0 \\
& & \mathbb{Z}^k & \rightarrow & \text{Pic}(\tilde{V}(\xi)) & \rightarrow & \text{Cl}(\tilde{R}) & \rightarrow & 0 \\
\end{array}
\]

Where the map $\text{Cl}(R) \rightarrow \text{Cl}(\tilde{R})$ is the natural completion map, and the map $\text{Pic}(X) \rightarrow \text{Pic}(\tilde{V}(\xi))$ is the pullback along the projection map $\tilde{V}(\xi) \rightarrow X$.

**Proof.** Let $U := Y \setminus Z$ and $\tilde{U} := \tilde{Y} \setminus \tilde{Z}$, and note that $p$ and $\tilde{p}$ induce isomorphisms $C \setminus E \cong U$ and $\tilde{C} \setminus \tilde{E} \cong \tilde{U}$. Thus we will also regard $U$ and $\tilde{U}$ as being open subschemes of $C$ and $\tilde{C}$. As in [Dan68, Lemma 3] we get that $U$ and $\tilde{U}$ are both regular.

Now as $Z \subseteq Y$ and $\tilde{Z} \subseteq \tilde{Y}$ have codimension at least two, the restriction maps $\text{Cl}(R) = \text{Cl}(Y) \rightarrow \text{Cl}(U)$ and $\text{Cl}(\tilde{R}) = \text{Cl}(\tilde{Y}) \rightarrow \text{Cl}(\tilde{U})$ are isomorphisms.

Now as each $E_j \subseteq C$ is an irreducible subvariety of codimension 1, and $E = E_1 \cup E_2 \cup \cdots \cup E_k$, we get the the restriction map $\text{Cl}(C) \rightarrow \text{Cl}(C \setminus E) = \text{Cl}(U)$ is a surjection, with kernel equal to the $\mathbb{Z}$-span of $[E_1], \ldots, [E_k]$ (cf [Har77, Proposition II.6.5]).

We claim that $[E_1], \ldots, [E_k] \in \text{Cl}(C)$ are $\mathbb{Z}$-linearly independent. Assume not. Then there exists some non-unit rational function $g$ on $C$ for which $\text{div} \ g = n_1[E_1] + \cdots + n_k[E_k]$, and so in particular, $\text{supp} \ g \subseteq E$. But then writing $g = p'(g')$ for some rational function on $Y$, we get that $\text{supp} \ g' \subseteq p(E) = Z$, which implies that $g'$, and hence $g$, is a unit as $Z \subseteq Y$ has codimension at least two, a contradiction.

Thus we have an exact sequence $0 \rightarrow \mathbb{Z}^k \rightarrow \text{Cl}(C) \rightarrow \text{Cl}(R) \rightarrow 0$.

Similarly we have a surjection $\text{Cl}(\tilde{C}) \rightarrow \text{Cl}(\tilde{R})$ with kernel spanned by $[\tilde{E}_1], \ldots, [\tilde{E}_k] \in \text{Cl}(\tilde{C})$, which are also $\mathbb{Z}$-linearly independent. This gives the exact sequence $0 \rightarrow \mathbb{Z}^k \rightarrow \text{Cl}(\tilde{C}) \rightarrow \text{Cl}(\tilde{R}) \rightarrow 0$.

It remains to give isomorphisms $\text{Pic}(X) \cong \text{Cl}(C)$ and $\text{Pic}(\tilde{V}(\xi)) \cong \text{Cl}(\tilde{C})$ compatible with the other maps.

First, as $C$ and $\tilde{C}$ are locally factorial, we get that $\text{Cl}(C) \cong \text{Pic}(C)$ and $\text{Cl}(\tilde{C}) \cong \text{Pic}(\tilde{C})$. By [Dan68, Proposition 3], the zero section $X \hookrightarrow V(\xi)$ gives an isomorphism $\text{Pic}(V(\xi)) \cong \text{Pic}(X)$, so by Lemma 3.20, $\text{Pic}(X) \cong \text{Pic}(V(\xi)) \cong \text{Pic}(C) \cong \text{Cl}(C)$.
Now as \( \hat{R} \) is an adic Noetherian ring with ideal of definition \( \hat{m}_R \), the morphism \( \tilde{p} : \tilde{C} \to \tilde{Y} = \text{Spec} \hat{R} \) is projective, and \( \tilde{C} \) is the completion of \( \tilde{C} \) along \( X = \tilde{p}^{-1}(\{m_R\}) \), the argument of [Dan68, Proposition 4] implies that \( \text{Pic}(\tilde{C}) \cong \text{Pic}(\tilde{C}) \) is an isomorphism.

Thus Lemma 3.20 gives \( \text{Pic}(\tilde{V}(\xi)) \cong \text{Pic}(\tilde{C}) \cong \text{Pic}(\tilde{C}) \), establishing the desired commutative diagram. \( \square \)

Thus it will suffice to show that the map \( \text{Pic}(X) \to \text{Pic}(\tilde{V}(\xi)) \) induced by the projection \( \tilde{V}(\xi) \to X \) is an isomorphism.

Now write \( C_n = \text{Spec}_X \left( \bigoplus_{i=0}^{n} \xi^{\otimes i} \right) \) (where \( \text{Spec}_X \) denotes the relative Spec over \( X \)), so that \( \tilde{V}(\xi) = \varprojlim_n C_n \). As in [Dan68, Proposition 5] we have \( \text{Pic}(\tilde{V}(\xi)) \cong \varprojlim_n \text{Pic}(C_n) \).

Now for each \( n \), let \( \text{pr}_n : C_n \to X \) be the projection, and let \( i_n : X \to C_n \) be the zero section. Note that we can canonically have \( C_0 = X \) and \( i_0 \) and \( \text{pr}_0 \) are just the identity map.

We have that \( \text{pr}_n \circ i_n = \text{id}_X \) and so \( i_n^* \circ \text{pr}_n^* = \text{id}_{\text{Pic}(X)} \). Hence \( \text{pr}_n^* : \text{Pic}(X) \to \text{Pic}(C_n) \) is an injection (and in fact, \( \text{Pic}(X) \) is a direct summand of \( \text{Pic}(C_n) \)). It follows that the map \( \text{pr}^* = (\text{pr}_n^*) : \text{Pic}(X) \to \varprojlim_n \text{Pic}(C_n) \cong \text{Pic}(\tilde{V}(\xi)) \) is injective. In particular this means that \( \text{Cl}(R) \to \text{Cl}(\hat{R}) \) is injective.

Now for each \( n \) we have \( \text{Pic}(C_n) = H^1(X, \mathcal{O}_{C_n}^*) \). As in [Dan68, Section 3], we consider the exact sequence of sheaves on \( X \):

\[
0 \to \xi^{\otimes (n+1)} \to \mathcal{O}_{C_{n+1}}^* \to \mathcal{O}_{C_n}^* \to 1,
\]

where the first map sends \( s \in \Gamma(W, \xi^{\otimes (n+1)}) \) to \( 1 + s \in \Gamma(W, \mathcal{O}_{C_{n+1}}^*) \). Then the long exact sequence of cohomology gives an exact sequence:

\[
H^1(X, \xi^{\otimes (n+1)}) \to \text{Pic}(C_{n+1}) \to \text{Pic}(C_n) \to H^2(X, \xi^{\otimes (n+1)}).
\]

We now claim that \( H^d(X, \xi^{\otimes i}) = 0 \) for all \( d = 1, 2 \) and \( i \geq 0 \). First note that

\[
\xi^{\otimes i} = \left( \bigoplus_{j=1}^{k} \pi_j^* \mathcal{L}_j \right)^{\otimes i} = \bigoplus_{i_1 + \cdots + i_k = i, i_1, \ldots, i_k \geq 0} \left( \pi_1^{i_1} \mathcal{L}_1^{\otimes i_1} \otimes \cdots \otimes \pi_k^{i_k} \mathcal{L}_k^{\otimes i_k} \right) = \bigoplus_{i_1 + \cdots + i_k = i, i_1, \ldots, i_k \geq 0} \mathcal{L}_1^{\otimes i_1} \otimes \cdots \otimes \mathcal{L}_k^{\otimes i_k}
\]

but now for any \( i_1, \ldots, i_k \geq 0 \) and any \( d = 1, 2 \) we get:

\[
H^d(X, \mathcal{L}_1^{\otimes i_1} \otimes \cdots \otimes \mathcal{L}_k^{\otimes i_k}) = H^d(\mathcal{V}_1 \times \cdots \times \mathcal{V}_k, \mathcal{L}_1^{\otimes i_1} \otimes \cdots \otimes \mathcal{L}_k^{\otimes i_k})
\]

\[
= \bigoplus_{d_1 + \cdots + d_k = d} \bigotimes_{j=1}^{k} H^d_j(\mathcal{V}_j, \mathcal{L}_j^{\otimes i_j}) = 0,
\]

\[32\]
since for any \( k \)-tuple \( (d_1, \ldots, d_k) \) with \( d_1 + \cdots + d_k = d \in \{1, 2\} \) and \( d_1, \ldots, d_k \geq 0 \), there must be some index \( j \) for which \( d_j \in \{1, 2\} \), and so \( H^{d_j}(V_j, \mathcal{L}^{\otimes j}) = 0 \) by assumption.

Thus for any \( n \geq 0 \), we indeed get that \( H^1(X, \xi^{\otimes (n+1)}) = H^2(X, \xi^{\otimes (n+1)}) = 0 \), and so we have \( \text{Pic}(C_{n+1}) \cong \text{Pic}(C_n) \). Thus as \( \text{pr}_n^* : \text{Pic}(X) \to \text{Pic}(C_n) \) is an isomorphism, it follows by induction that \( \text{pr}_n^* : \text{Pic}(X) \to \text{Pic}(C_n) \) is an isomorphism for all \( n \), and so \( \text{pr} : \text{Pic}(X) \to \lim_{\longrightarrow} \text{Pic}(C_n) = \text{Pic}(\hat{V}(\xi)) \) is an isomorphism.

Hence the completion map \( \text{Cl}(R) \to \text{Cl}(\hat{R}) \) is indeed an isomorphism, completing the proof. \( \square \)

So indeed, \( \text{Cl}(\mathcal{R}) \to \text{Cl}(\mathcal{R}_\infty) \) is an isomorphism. As noted above, this completes the proof of Theorem 3.3, and hence of Theorem 3.1.

4 The construction of \( M_\infty \)

From now on assume that \( \bar{\rho} : G_F \to \text{GL}_2(\mathbb{F}) \) satisfies condition (4) of Theorem 1.1 (i.e. the “Taylor-Wiles” condition). The goal of this section is to construct a module \( M_\infty \) over \( R_\infty \) satisfying the conditions of Theorem 3.1.

We shall construct \( M_\infty \) by applying the Taylor–Wiles–Kisin patching method [Wil95, TW95, Kis09b] to a natural system of modules over the rings \( T_{D}(K) \). For convenience we will follow the “Ultrapatching” construction introduced by Scholze in [Sch18]. The primary advantage to doing this is that Scholze’s construction is somewhat more “natural” than the classical construction, and thus it will be easier to show that \( M_\infty \) satisfies additional properties (in our case, that it is self-dual).

4.1 Ultrapatching

In this subsection, we briefly recall Scholze’s construction (while introducing our own notation).

From now on, fix a nonprincipal ultrafilter \( \mathcal{F} \) on the natural numbers \( \mathbb{N} \) (it is well known that such an \( \mathcal{F} \) must exist, provided we assume the axiom of choice). For convenience, we will say that a property \( P(n) \) holds for \( \mathcal{F} \)-many \( i \) if there is some \( I \in \mathcal{F} \) such that \( P(i) \) holds for all \( i \in I \).

For any collection of sets \( \mathcal{A} = \{ A_n \}_{n \geq 1} \), we define their \textit{ultraproduct} to be the quotient

\[
\mathcal{U}(\mathcal{A}) := \left( \prod_{n=1}^{\infty} A_n \right) / \sim
\]

where we define the equivalence relation \( \sim \) by \( (a_n)_n \sim (a'_n)_n \) if \( a_i = a'_i \) for \( \mathcal{F} \)-many \( i \).

If the \( A_n \)'s are sets with an algebraic structure (eg. groups, rings, \( R \)-modules, \( R \)-algebras, etc.) then \( \mathcal{U}(\mathcal{A}) \) naturally inherits the same structure.
Also if each $A_n$ is a finite set, and the cardinalities of the $A_n$’s are bounded (this is the only situation we will consider in this paper), then $\mathcal{U}(\mathcal{A})$ is also a finite set and there are bijections $\mathcal{U}(\mathcal{A}) \sim A_i$ for $\mathfrak{F}$-many $i$. Moreover if the $A_n$’s are sets with an algebraic structure, such that there are only finitely many distinct isomorphism classes appearing in $\{A_n\}_{n \geq 1}$ (which happens automatically if the structure is defined by finitely many operations, eg. groups, rings or $R$-modules or $R$-algebras over a finite ring $R$) then these bijections may be taken to be isomorphisms. This is merely because our conditions imply that there is some $A$ such that $A \cong A_i$ for $\mathfrak{F}$-many $i$ and hence $\mathcal{U}(\mathcal{A})$ is isomorphic to the “constant” ultraproduct $\mathcal{U}(\{A\}_{n \geq 1})$ which is easily seen to be isomorphic to $A$, provided that $A$ is finite.

Lastly, in the case when each $A_n$ is a module over a finite local ring $R$, there is a simple algebraic description of $\mathcal{U}(\mathcal{A})$. Specifically, the ring $\mathcal{R} = \prod_{n=1}^{\infty} R$ contains a unique maximal ideal $\mathfrak{m}_\mathcal{R} \in \text{Spec } \mathcal{R}$ for which $\mathcal{R}/\mathfrak{m}_\mathcal{R} \cong R$ and $\prod_{n=1}^{\infty} A_n \cong \mathcal{U}(\mathcal{A})$ as $R$-modules. This shows that $\mathcal{U}(-)$ is a particularly well-behaved functor in our situation. In particular, it is exact.

For the rest of this section, fix a power series ring $S_\infty = \mathcal{O}[\![z_1, \ldots, z_t]\!]$ and consider the ideal $\mathfrak{n} = (z_1, \ldots, z_n)$.

We can now make our main definitions:

**Definition 4.1.** Let $\mathcal{M} = \{M_n\}_{n \geq 1}$ be a sequence of finite type $S_\infty$-modules.

- We say that $\mathcal{M}$ is a weak patching system if the $S_\infty$-ranks of the $M_n$’s are uniformly bounded.
- We say that $\mathcal{M}$ is a patching system if it is a weak patching system, and for any open ideal $\mathfrak{a} \subseteq S_\infty$, we have $\text{Ann}_{S_\infty}(M_i) \subseteq \mathfrak{a}$ for all but finitely many $n$.
- We say that $\mathcal{M}$ is free if $M_n$ is free over $S_\infty/\text{Ann}_{S_\infty}(M_n)$ for all but finitely many $n$.

Furthermore, assume that $\mathcal{R} = \{R_n\}_{n \geq 1}$ is a sequence of finite type $S_\infty$-algebras.

- We say that $\mathcal{R} = \{R_n\}_{n \geq 1}$ is a (weak) patching algebra, if it is a (weak) patching system.
- If $M_n$ is an $R_n$-module (viewed as an $S_\infty$-module via the $S_\infty$-algebra structure on $R_n$) for all $n$ we say that $\mathcal{M} = \{M_n\}_{n \geq 1}$ is a (weak) patching $\mathcal{R}$-module if it is a (weak) patching system.

Now for any weak-patching system $\mathcal{M}$, we define its patched module to be the $S_\infty$-module

$$\mathcal{P}(\mathcal{M}) := \lim_{\mathfrak{a}} \mathcal{U}(\mathcal{M}/\mathfrak{a}),$$

where the inverse limit is taken over all open ideals of $S_\infty$.

If $\mathcal{R}$ is a (weak) patching algebra and $\mathcal{M}$ is a (weak) patching $\mathcal{R}$-module, then $\mathcal{P}(\mathcal{R})$ inherits a natural $S_\infty$-algebra structure, and $\mathcal{P}(\mathcal{M})$ inherits a natural $\mathcal{P}(\mathcal{R})$-module structure.

In the above definition, the ultraproduct essentially plays the role of pigeonhole principal in the classical Taylor-Wiles construction, with the simplification that is is not necessary to explicitly
define a “patching datum” before making the construction. Indeed, if one were to define patching data for the $M_n/a$’s (essentially, imposing extra structure on each of the modules $M_n/a$) then the machinery of ultraproducts would ensure that the patching data for $U(\mathcal{M}/a)$ would agree with that of $M_n/a$ for infinitely many $n$. It is thus easy to see that our definition agrees with the classical construction (cf. [Sch18]).

Thus the standard results about patching (cf [Kis09b]) may be rephrased as follows:

**Theorem 4.2.** Let $\mathcal{R}$ be a weak patching algebra, and let $\mathcal{M}$ be a free patching $\mathcal{R}$-module. Then:

1. $\mathcal{P}(\mathcal{R})$ is a finitely generated free $S_\infty$-module. $\mathcal{P}(\mathcal{M})$ is a finitely generated free $S_\infty$-module.
2. The structure map $S_\infty \to \mathcal{P}(\mathcal{R})$ (defining the $S_\infty$-algebra structure) is injective, and thus $\dim \mathcal{P}(\mathcal{R}) = \dim S_\infty$.
3. The module $\mathcal{P}(\mathcal{M})$ is maximal Cohen–Macaulay over $\mathcal{P}(\mathcal{R})$. $(\lambda, z_1, \ldots, z_\ell)$ is a regular sequence for $\mathcal{P}(\mathcal{M})$.
4. Let $n = (z_1, \ldots, z_\ell) \subseteq S_\infty$, as above. Let $R_0$ be a finite type $O$-algebra, and let $M_0$ be a finitely generated $R_0$-module. If, for each $n \geq 1$, there are isomorphisms $R_n/n \cong R_0$ of $O$-algebras and $M_n/n \cong M_0$ of $R_n/n \cong R_0$-modules, then we have $\mathcal{P}(\mathcal{R})/n \cong R_0$ as $O$-algebras and $\mathcal{P}(\mathcal{M})/n \cong M_0$ as $\mathcal{P}(\mathcal{R})/n \cong R_0$-modules.

From the set up of Theorem 4.2 there is very little we can directly conclude about the ring $\mathcal{P}(\mathcal{R})$. However in practice one generally takes the rings $R_n$ to be quotients of a fixed ring $R_\infty$ (which in our case will be a result of Lemma 2.5) of the same dimension as $S_\infty$ (and thus as $\mathcal{P}(\mathcal{R})$). Thus we define a cover of a weak patching algebra $\mathcal{R} = \{R_n\}_{n \geq 1}$ to be a pair $(R_\infty, \{\varphi_n\}_{n \geq 1})$ (which we will denote by $R_\infty$ when the $\varphi_n$’s are clear from context), where $R_\infty$ is a complete, topologically finitely generated $O$-algebra of Krull dimension $\dim S_\infty$ and $\varphi_n : R_\infty \to R_n$ is a surjective $O$-algebra homomorphism for each $n$. We have the following:

**Theorem 4.3.** If $(R_\infty, \{\varphi_n\})$ is a cover of a weak patching algebra $\mathcal{R}$, then the $\varphi_n$’s induce a natural continuous surjection $\varphi_\infty : R_\infty \to \mathcal{P}(\mathcal{R})$. If $R_\infty$ is a domain then $\varphi_\infty$ is an isomorphism.

**Proof.** The $\varphi_n$’s induce a continuous map $\Phi = \prod_{n \geq 1} \varphi_n : R_\infty \to \prod_{n \geq 1} R_n$, and thus induce continuous maps

$$\Phi_a : R_\infty \to \prod_{n \geq 1} R_n \to \prod_{n \geq 1} (R_n/a) \to U(\mathcal{R}/a)$$

for all open $a \subseteq S_\infty$. Hence they indeed induce a continuous map

$$\varphi_\infty = (\Phi_a)_a : R_\infty \to \lim_{a} U(\mathcal{R}/a) = \mathcal{P}(\mathcal{R}).$$

Now as $R_\infty$ is complete and topologically finitely generated, it is compact, and thus $\varphi_\infty(R_\infty) \subseteq \mathcal{P}(\mathcal{R})$ is closed. So to show that $\varphi_\infty$ is surjective, it suffices to show that $\varphi_\infty(R_\infty)$ is dense, or equivalently that each $\Phi_a$ is surjective.

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Now for any \( n \) and any open \( a \subseteq R_n/a \) is a finite set with the structure of a continuous \( R_\infty \) algebra (defined by the continuous surjection \( \varphi_n : R_\infty \rightarrow R_n \rightarrow R_n/a \)) and the cardinalities of the \( R_n/a \)’s are bounded. As noted above, this implies that \( \mathcal{U}(\mathcal{R}/a) \) also has the structure of an \( R_\infty \)-algebra (which is just the structure induced by \( \Phi_a \)). As \( R_\infty \) is topologically finitely generated, there are only finitely many distinct isomorphism classes of \( R_\infty \)-algebras in \( \{ R_n/a \}_{n \geq 1} \). By the above discussion of ultraproducts, this implies that \( R_i/a \cong \mathcal{U}(\mathcal{R}/a) \) as \( R_\infty \)-algebras for \( \mathfrak{F} \)-many \( i \).

But now taking any such \( i \), as the structure map \( R_\infty \rightarrow R_i/a \) is surjective, and so the structure map \( \Phi_a : R_\infty \rightarrow \mathcal{U}(\mathcal{R}/a) \) is as well.

The final claim simply follows by noting that if \( R_\infty \) is a domain and \( \varphi_\infty \) is not injective, then \( \mathcal{P}(\mathcal{R}) \cong R_\infty/\ker \varphi_\infty \) would have Krull dimension strictly smaller than \( R_\infty \), contradicting our assumption that \( \dim R_\infty = \dim S_\infty = \dim \mathcal{P}(\mathcal{R}) \).

In order to construct the desired module \( M_\infty \) over \( R_\infty \) satisfying the conditions of Theorem 3.1, we will construct a weak patching algebra \( \mathcal{R}^\mathcal{R} \) covered by \( R_\infty \), and a free patching \( \mathcal{R}^\mathcal{R} \)-module \( \mathcal{M}^\mathcal{R} \), and then define \( M_\infty = \mathcal{P}(\mathcal{M}^\mathcal{R}) \).

### 4.2 Spaces of automorphic forms

In this section, we will construct the spaces of automorphic forms \( M(K) \) and \( M_{\mathfrak{f}}(K) \) that will be used in Section 4.3 to construct the patching system \( \mathcal{M}^\mathcal{R} \), producing \( M_\infty \).

Recall that \( \varphi : G_F \rightarrow \GL_2(\mathbb{F}_L) \) is assumed to be a Galois representation satisfying all of the conditions of Theorem 1.1. In particular \( \mathcal{K}^D(\varphi) \neq \emptyset \), so that \( \bar{\varphi} = \bar{\varphi}_m \) for some \( K \in \mathcal{K}^D(\varphi) \) and some \( m \subseteq T^D(K) \). By enlarging \( \mathcal{O} \) if necessary, assume that \( T^D(K)/m = \mathbb{F} \) and \( \mathbb{F} \) contains all eigenvalues of \( \bar{\varphi}(\sigma) \) for all \( \sigma \in G_F \).

Since the results of Theorems 1.1 and 1.2 are known classically in the case when \( F = \mathbb{Q} \) and \( D = \GL_2 \), we will exclude this case for convenience. Thus we will assume that \( D(F) \) is a division algebra.

For any \( K \in \mathcal{K}^D(\varphi) \), define \( M(K) := S^D(K)_m^\vee \) if \( D \) is totally definite and

\[
M(K) := \Hom_{R,F,S(\varphi)[G_F]}(\rho^{\text{univ}}, S^D(K)_m^\vee)
\]

if \( D \) is indefinite. Note that this definition depends only on the \( T^D(K)_m \)-module structure of \( S^D(K)_m^\vee \), and not on the specific choice of \( S \) in \( R_{F,S}(\varphi) \). Give \( M(K) \) its natural \( T^D(K)_m \)-module structure.

**Remark.** The purpose of the definition of \( M(K) \) in the indefinite case is to “factor out” the Galois action on \( S^D(K)^\vee \). This construction was described by Carayol in [Car94]. As in [Car94] we have that the natural evaluation map \( M(K) \otimes_{R,F,S(\varphi)} \rho^{\text{univ}} \rightarrow S^D(K)_m^\vee \) is an isomorphism, and so \( S^D(K)_m^\vee \cong M(K)^{\otimes 2} \) as \( T^D(K)_m \)-modules.

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If we did not do this, and only worked with \( S^D(K)^\vee_m \), then the module \( M_\infty \) we will construct would have generic rank 2 instead of generic rank 1, and so we would not be able to directly apply Theorem 3.1.

Note that it follows from the definitions that \( \dim_F M(K)/m = \nu_F(K) \) for all \( K \).

For technical reasons (related to the proof of Lemma 2.5) we cannot directly apply the patching construction to the modules \( M(K) \). Instead, it will be necessary to introduce “fixed-determinant” versions of these spaces, \( M_\psi(K) \).

We now make the following definition: A level \( K \subseteq D^\times(\mathbb{A}_{F,f}) \) is sufficiently small if for all \( t \in D^\times(\mathbb{A}_{F,f}) \) we have \( K \mathbb{A}_{F,f}^\times \cap (t^{-1}D^\times(F)t) = F^\times \). This is condition (2.1.2) in [Kis09a]. Note that this implies that the Shimura variety \( X_K \) does not contain any elliptic points.

The importance of considering sufficiently small levels is the following standard lemma:

**Lemma 4.4.** Let \( K \subseteq D^\times(\mathbb{A}_{F,f}) \) be a level, and let \( K' \triangleleft K \) be a level which is a normal subgroup of \( K \). Consider a finite subgroup \( G \subseteq K \mathbb{A}_{F,f}^\times/K' F^\times \), and let \( \mathcal{O}[G] \) be its group ring and \( a_G \subset \mathcal{O}[G] \) be the augmentation ideal. If \( K \) is sufficiently small then:

1. If \( D \) is totally definite (resp. indefinite) then \( G \) acts freely on the double quotient \( D^\times(F)\backslash D^\times(\mathbb{A}_{F,f})/K' \) (resp. the Riemann surface \( X^D(K') = D^\times(F)\backslash (D^\times(\mathbb{A}_{F,f}) \times \mathcal{H})/K' \)) by right multiplication.
2. \( S^D(K')^\vee \) is a finite projective \( \mathcal{O}[G] \)-module.
3. If \( G = K F^\times/K' F^\times \) then the operator \( \sum g \in G \) induces an isomorphism \( S^D(K')^\vee/a_G S^D(K')^\vee \cong S^D(K)^\vee \).

**Proof.** This essentially follows from the argument of [Kis09a, Lemma (2.1.4)]. See also the proof of [dS97, Proposition 14] for the argument that (2) and (3) follow from (1) in the indefinite case. \( \square \)

This lemma will allow us to construct the desired free patching system \( \mathcal{M}^\square \). However, in order to use this lemma it will be necessary to first restrict our attention to sufficiently small levels \( K \). First by the conditions on \( \mathfrak{p} \) and [DDT97, Lemma 4.11] we may pick a prime \( w \notin \Sigma^D_F \) satisfying

- \( \text{Nm}(w) \not\equiv 1 \pmod{\ell} \)
- The ratio of the eigenvalues of \( \mathfrak{p}(\text{Frob}_w) \) is not equal to \( \text{Nm}(w)^{\pm 1} \) in \( \mathbb{F}_\ell^\times \).
- For any nontrivial root of unity \( \zeta \) for which \( [F(\zeta) : F] \leq 2, \zeta + \zeta^{-1} \not\equiv 2 \pmod{w} \).

Define

\[
U_w = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_{F,w}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{w} \right\} \subseteq \text{GL}_2(F_w)
\]

\[
U_w^- = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_{F,w}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{w} \right\} \subseteq \text{GL}_2(F_w)
\]

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Let $K_0$ (resp. $K_0^-$) be the preimage of $U_w$ (resp. $U_w^-$) under the map $K^{\min} \rightarrow D^\times(\mathbb{A}_{F,f}) \rightarrow \text{GL}_2(F_w)$.

We then have the following:

**Lemma 4.5.** $K_0$ is sufficiently small, and we have compatible isomorphisms $\mathbb{T}^D(K^{\min})_m \cong \mathbb{T}^D(K_0)_m$, $S^D(K^{\min})_m \cong S^D(K_0)_m$ and $M(K^{\min}) \cong M(K_0)$.

**Proof.** The fact that $K_0$ is sufficiently small follows easily from last hypothesis on $w$ (cf [Kis09b, (2.1.1)]). As in [DDT97, Section 4.3], the first two conditions on $w$ imply that $w$ is not a level-raising prime for $\overline{\mathfrak{p}}$ and so we obtain natural isomorphisms $\mathbb{T}^D(K^{\min})_m \cong \mathbb{T}^D(K_0^-)_m \cong \mathbb{T}^D(K_0)_m$. By the definition of $M(K)$, the isomorphism $M(K^{\min}) \cong M(K_0)$ will follow from $S^D(K^{\min})_m \cong S^D(K_0)_m$, so it suffices to prove this isomorphism.

It follows from the argument of [Tay06, Lemma 2.2] that there is an isomorphism $S^D(K^{\min})_m \cong S^D(K_0^-)_m$. Now the argument of [BDJ10, Lemma 4.11] implies that the map $S^D(K_0^-)_m \rightarrow S^D(K_0)_m$ is an isomorphism (as the assumptions that $\ell > 2$ and $\mathfrak{p}|_{G_F(\zeta)}$ is absolutely irreducible imply that $\mathfrak{p}$ is not “badly dihedral”, in the sense defined in that argument). Finally, as $K_0/K_0^- \cong ((\mathcal{O}_F/w)^\times)^2$ has prime to $\ell$ order, we get that $S^D(K_0^-)_m \cong S^D(K_0)_m$, giving the desired isomorphism. \hfill $\square$

It now follows that $\nu_{\mathfrak{p}}(K^{\min}) = \nu_{\mathfrak{p}}(K_0)$. We will now restrict our attention to levels contained in $K_0$.

For any level $K \subseteq K_0$, let $C_K := F^\times \backslash \mathbb{A}_{F,f}^\times/(K \cap \mathbb{A}_{F,f}^\times)$ denote the image of $\mathbb{A}_{F,f}^\times$ in the double quotient $D^\times(F)/D^\times(\mathbb{A}_{F,f})/K$. Note that this is a finite abelian group. For any finite place $v$ of $F$, let $\varpi_v$ denote the image of the uniformizer $\varpi_v \in F_v^\times \subseteq \mathbb{A}_{F,f}^\times$ in $C_K$.

By the definition of $S_v$, we see that $\varpi_v$ acts on $S^D(K)^\vee$ as $S_v$ for all $v \notin S$, and so we may identify $\mathcal{O}[C_K]$ with a subring of $\mathbb{T}^D(K)$. Specifically, it is the $\mathcal{O}$-subalgebra generated by the Hecke operators $S_v$ for $v \notin S$.

Now the action of $C_K$ on $D^\times \backslash D^\times(\mathbb{A}_{F,f})/K$ induces an action of $\mathcal{O}[C_K]$ on $M(K)$. By Lemma 4.4 (with $K' = K$ and $G = C_K \hookrightarrow K^\times \mathbb{A}_{F,f}^\times/K$) $S^D(K)^\vee$ is a finite projective (and hence free) $\mathcal{O}[C_K]$-module. Let $m' = m \cap \mathcal{O}[C_K]$, so that $m'$ is a maximal ideal of $\mathcal{O}[C_K]$. It follows that $S^D(K)_m'$ is a finite free $\mathcal{O}[C_K]_{m'}$-module.

Let $C_{K,\ell} \leq C_K$ be the Sylow $\ell$-subgroup. Since $C_K$ is abelian, we have $C_K \cong C_{K,\ell} \times (C_K/C_{K,\ell})$ and so $\mathcal{O}[C_K] \cong \mathcal{O}[C_{K,\ell}] \otimes_{\mathcal{O}} \mathcal{O}[C_K/C_{K,\ell}]$. Now as $C_K/C_{K,\ell}$ has prime to $\ell$ order, by enlarging $\mathcal{O}$ if necessary, we may assume that $\mathcal{O}[C_K/C_{K,\ell}] \cong \mathcal{O}^{\oplus \#(C_K/C_{K,\ell})}$ as an $\mathcal{O}$-algebra, and so $\mathcal{O}[C_K] \cong \mathcal{O}^{\oplus \#(C_K/C_{K,\ell})}$. But now as $\mathcal{O}[C_{K,\ell}]$ is a complete local $\mathcal{O}$-algebra (as $\mathcal{O}[\mathbb{Z}/\ell^n\mathbb{Z}] \cong \mathcal{O}[x]/((1 + x)^{\ell^n} − 1)$ is for any $n$, and $C_{K,\ell}$ is a finite abelian $\ell$-group), it follows that $\mathcal{O}[C_K]_{m'} \cong \mathcal{O}[C_{K,\ell}]$ for

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8Note that this proof does not rely on Taylor’s assumption that $\text{Nm}(w) = 1$, only on the assumption that the ratio of the eigenvalues of $\overline{\mathfrak{p}}(\text{Frob}_w)$ is not $\text{Nm}(w)^{\pm 1}$. Also by the definition of $U_w$, no lift of $\overline{\mathfrak{p}}$ occurring in $S^D(K_0^-)$ can have determinant ramified at $w$, and so the fact that we have not yet fixed determinants does not affect the argument.
any maximal ideal $m'$. Hence there is an embedding $\mathcal{O}[C_{K,\ell}] \hookrightarrow T^D(K)_m$ which makes $S^D(K)_m$ into a finite free $\mathcal{O}[C_{K,\ell}]$-module.

It follows that $M(K)$ is also a finite free $\mathcal{O}[C_{K,\ell}]$-module. Indeed, this is simply by definition in the case when $D$ is definite. If $D$ is indefinite, this follows from the fact that $M(K)\otimes 2 \cong S^D(K)_m$ is free, and direct summands of free $\mathcal{O}[C_{K,\ell}]$-modules are projective and hence free.

Now fix a character $\psi : G_F \to \mathcal{O}^\times$ for which $m$ is in the support of $T^D_{\psi}(K^{\mathrm{min}})$. For any level $K \subseteq K_0$, define an ideal $\mathfrak{I}_\psi := (Nm(v)[\overline{\rho}_v] - \psi(\text{Frob}_v)|v \not\in S) \subseteq \mathcal{O}[C_{K,\ell}]$. As $m$ is also in the support of $T^D_{\psi}(K)$, it follows that $\mathfrak{I}_\psi$ contained in the kernel of some map $T^D_{\psi}(K) \to \mathcal{O}$ (corresponding to some lift of $\rho : G_F \to \text{GL}_2(\mathcal{O})$ of $\overline{\rho}$ which is modular of level $K$ and has $\det \rho = \psi$), and so we can deduce that $\mathcal{O}[C_{K,\ell}]/\mathfrak{I}_\psi \cong \mathcal{O}$. We may now define $M_\psi(K) := M(K)/\mathfrak{I}_\psi M(K)$. It follows that $M_\psi(K)$ is a finite free $\mathcal{O}$-module. Moreover, by definition it follows that $T^D_{\psi}(K)_m$ is exactly the image of $T^D(K)_m$ in $\text{End}_{\mathcal{O}}(\mathcal{M}_\psi(K))$, and $M_\psi(K) = M(K) \otimes_{T^D(K)_m} T^D_{\psi}(K)_m$.

It is necessary to consider the modules $M_\psi(K)$ instead of $M(K)$, because the patching argument requires us to work with fixed-determinant deformation rings. Fortunately, as $\mathfrak{I}_\psi \subseteq m$, we get $\dim_{\mathbb{F}} M_\psi(K_0)/m = \dim M(K_0)/m = \nu_{\mathbb{F}}(K_0) = \nu_{\mathbb{F}}(K^{\mathrm{min}})$, and so considering the $M_\psi(K)$’s instead of the $M(K)$’s will still be sufficient to prove Theorem 1.1.

### 4.3 A Patching System Producing $M_\infty$

For the rest of this paper, we will take the ring $S_\infty$ from the Section 4.1 to be $\mathcal{O}[[y_1, \ldots, y_r, w_1, \ldots, w_j]]$, where $r$ is an in Lemma 2.5 and $j = 4|\Sigma^D_\ell| - 1$ is as in Section 2.2, and let $n = (y_1, \ldots, y_r, w_1, \ldots, w_j)$ as before. Note that $\dim S_\infty = r + j + 1 = \dim R_\infty$ by Lemma 2.5.

We will construct a weak patching algebra $\mathfrak{R}^\square$ covered by $R_\infty$ using the deformation rings $R_{F,S,\psi,\mathfrak{Q}_n}(\overline{\rho})$, and construct a free patching $\mathfrak{R}^\square$-module $\mathfrak{M}^\square$ using the spaces $M_\psi(K)$ constructed above. We then take $M_\infty := \mathfrak{P}(\mathfrak{M}^\square)$. By Theorems 4.2 and 4.3 it will then follow that $M_\infty$ is maximal Cohen–Macaulay over $R_\infty$. In Section 4.4, we will show that $M_\infty$ satisfies the remaining conditions of Theorem 3.1.

From now on, fix $S = \Sigma^D \cup \{w\}$, where $w$ is the prime chosen in Section 4.2 above, and fix a collection of sets of primes $Q = \{Q_n\}_{n \geq 1}$ satisfying the conclusion of Lemma 2.5. For any $n$, let $\Delta_n$ be the maximal $\ell$-power quotient of $\prod_{v \in Q_n} (\mathcal{O}_F/v)^\times$. Consider the ring $\Lambda_n := \mathcal{O}[\Delta_n]$, and note that:

$$\Lambda_n \cong \mathcal{O}[[y_1, \ldots, y_r]] / ((1 + y_1)^{e(n,1)} - 1, \ldots, (1 + y_r)^{e(n,r)} - 1)$$

where $\ell^{e(n,i)}$ is the $\ell$-part of $\text{Nm}(v) - 1 = \#(\mathcal{O}_F/v)^\times$, so that $e(n,i) \geq n$ by assumption. Let $a_n = (y_1, \ldots, y_r) \subseteq \Lambda_n$ be the augmentation ideal.

Also let $H_n = \ker \left( \prod_{v \in Q_n} (\mathcal{O}_F/v)^\times \to \Delta_n \right)$. For any finite place $v$ of $F$, consider the compact open
subgroups of $GL_2(O_{F,v})$:

$$\Gamma_0(v) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(O_{F,v}) \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod v \right\} \subseteq GL_2(O_{F,v})$$

$$\Gamma_1(v) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(v) \left| a \equiv d \mod v \right\} \subseteq \Gamma_0(v) \subseteq GL_2(O_{F,v})$$

Notice that $\Gamma_1(v) \subseteq \Gamma_0(v)$ and we have group isomorphisms

$$\Gamma_0(v)/\Gamma_1(v) \cong \Gamma_0(v)F^\times/\Gamma_1(v)F^\times \xrightarrow{\sim} (O_F/v)^\times$$

given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad^{-1} \mod v$. Now let $\gamma_H(Q_n) \subseteq \prod_{v \in Q_n} \Gamma_0(v)$ be the preimage of $\gamma_n \subseteq \prod_{v \in Q_n} (O_F/v)^\times$ under the map

$$\prod_{v \in Q_n} \Gamma_0(v) \twoheadrightarrow \prod_{v \in Q_n} \Gamma_0(v)/\Gamma_1(v) \cong \prod_{v \in Q_n} (O_F/v)^\times$$

finally let $K_n \subseteq K_0$ be the preimage of $\gamma_H(Q_n)$ under

$$K_0 \hookrightarrow \prod_{v \subseteq O_F} D^\times(O_{F,v}) \twoheadrightarrow \prod_{v \in Q_n} D^\times(O_{F,v}).$$

Now for any $n \geq 1$, any $v \in Q_n$ and any $\delta \in (O_F/v)^\times$, consider the double coset operators $U_v, \langle \delta \rangle_v : S^D(K_n) \rightarrow S^D(K_n)$ defined by

$$U_v = \left[ K \begin{pmatrix} \alpha_v & 0 \\ 0 & 1 \end{pmatrix} K \right], \quad \langle \delta \rangle_v = \left[ K \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} K \right],$$

where $d \in O_F$ is a lift of $\delta \in (O_F/v)^\times$. Note that $\langle \delta \rangle_v$ does not depend on the choice of $d$. In fact, if $\delta, \delta' \in (O_F/v)^\times$ have the same image under $(O_F/v)^\times \rightarrow \prod_{v \in Q_n} (O_F/v)^\times \rightarrow \Delta_n$, then $\langle \delta \rangle_v = \langle \delta' \rangle_v$. Define

$$T^D(K_n) := T^D(K_n) \left[ U_v, \langle \delta \rangle_v \left| v \in Q_n, \delta \in (O_F/v)^\times \right\} \subseteq \text{End}_O(S^D(K_n)),$$

and note that this is a commutative $O$-algebra extending $T^D(K_n)$, which is finite free as an $O$-module. Also for convenience set $T^D(K_0) := T^D(K_0)$.

Note that the double coset operators $U_v$ and $\langle \delta \rangle_v$ commute with the action of $T^D(K_n)$ and (in the case when $D$ is indefinite) $G_F$ on $S^D(K_n)$, and thus they descend to maps $U_v, \langle \delta \rangle_v : M_\psi(K_n) \rightarrow M_\psi(K_n)$. Let $T^D(K_n)_m$ denote the image of $T^D(K_n)$ in $\text{End}_O(M_\psi(K_n))$.

As in [Kis09b, (3.4.5)], $T^D(K_n)_m$ has $2^\#Q_n$ different maximal ideals (corresponding to the different possible choices of eigenvalue for $U_v$ for $v \in Q_n$). Fix any such maximal ideal $m_{Q_n} \subseteq T^D(K_n)_m$. Let
\[ T_n := (T^n_m(K_n)_{m=0})_{m=0^n} \] and \[ M_n := M^n_m(K_n)_{m=0^n} \] and \[ R_n := R^n_m(F,S\cup Q_n)_{m=0^n} \). Also define \( M_0 := M^n_m(K_0)_{m=0^n} \), \( T_0 := T^n_m(K_0)_{m=0^n} \) and \( R_0 := R^n_m(F,S)_{m=0^n} \) (note that \( M_n, T_n \) and \( R_n \) all have fixed determinant \( \psi \), but we are suppressing this in our notation).

We now have the following standard result (cf [dS97, DDT97, Kis09b], and also Lemma 4.4 above):

**Proposition 4.6.** For any \( n \geq 1 \), there is a surjective map \( R_n \to T_n \) giving \( M_n \) the structure of a \( T_n \)-module. There exists an embedding \( \Lambda_n \hookrightarrow R_n \) under which \( M_n \) is a finite rank free \( \Lambda_n \)-module. Moreover, we have \( R_n / \mathfrak{a}_n \cong R_0 \) and \( M_n / \mathfrak{a}_n \cong M_0 \) (so in particular, \( \text{rank}_{\Lambda_n} M_n = \text{rank}_O M_0 \)).

Now let \( R_n^\square = R^n_m(F,S\cup Q_n)_{m=0^n} \), and recall from Section 2.2 that \( R_n^\square = R_n[[w_1, \ldots, w_j]] \) for some integer \( j \). Using this, we may define framed versions of \( T_n \) and \( M_n \). Namely

\[
T_n^\square := R_n^\square \otimes_{R_n} T_n \cong T_n[[w_1, \ldots, w_j]]
\]

\[
M_n^\square := R_n^\square \otimes_{R_n} M_n \cong M_n[[w_1, \ldots, w_j]]
\]

so that \( M_n^\square \) inherits a natural \( T_n^\square \)-module structure, and we still have a surjective map \( R_n^\square \to T_n^\square \) (and so \( M_n^\square \) inherits a \( T_n^\square \)-module structure). Note that the ring structure of \( T_n^\square \) and the \( T_n^\square \)-module structure of \( M_n^\square \) do not depend on the choice of the set \( S \), and so we may define this without reference to a specific \( S \).

Also for any \( n \), consider the ring \( \Lambda_n^\square := \Lambda_n[[w_1, \ldots, w_j]] = \mathcal{O}[\Lambda_n][[w_1, \ldots, w_j]] \), which we will view as a quotient of the ring \( S_\infty = \mathcal{O}[[y_1, \ldots, y_r, w_1, \ldots, w_j]] \) from above.

Rewriting Proposition 4.6 in terms of the framed versions of \( R_n \) and \( M_n \), we get:

**Proposition 4.7.** There exists an embedding \( \Lambda_n^\square \hookrightarrow R_n^\square \) under which \( M_n^\square \) is a finite rank free \( \Lambda_n^\square \)-module. Moreover, we have \( R_n^\square / \mathfrak{n} \cong R_0 \) and \( M_n^\square / \mathfrak{n} \cong M_0 \) (so in particular, \( \text{rank}_{S_\infty} M_n^\square = \text{rank}_{\Lambda_n^\square} M_n^\square = \text{rank}_O M_0 \)).

So in particular, the rings \( R_n^\square \) are \( S_\infty \)-algebras and the modules \( M_n^\square \) are \( S_\infty \)-modules.

Lastly, for any \( n \geq 0 \), define \( R_n^{\square,D} = R_n^{\square,D}_m(F,S\cup Q_n)_{m=0^n} \) and \( R_n^D = R_n^{D,F,S\cup Q_n} \) (where we take \( Q_0 = 0 \)).

Note that the actions of \( R_n^\square \) and \( R_0 \) on \( M_n^\square \) and \( M_0 \) factor through \( R_n^{\square,D} \) and \( R_0^D \), respectively.

Temporarily writing \( R_n^{\square,D}_\Sigma,\Sigma,\epsilon = R_n^{\square,D}_\Sigma,\epsilon / J \), so that \( R_n^{\square,D} = R_n^{\square,D} / J \) for all \( n \geq 0 \), we also see that

\[
R_n^{\square,D} / \mathfrak{n} \cong (R_n^D / \mathfrak{n}) / J \cong (R_0^D / J) / (w_1, \ldots, w_j) \cong R_0^{\square,D} / (w_1, \ldots, w_j) \cong R_0^D.
\]

Now we claim that the collections \( \mathcal{R}_n^{\square,D} := \{ R_n^{\square,D} \}_{n \geq 1} \) and \( \mathcal{M}_n := \{ M_n^\square \}_{n \geq 1} \) satisfy the necessary conditions to apply Theorem 4.2. Specifically we have:

**Lemma 4.8.** \( \mathcal{R}_n^{\square,D} \) is a weak patching algebra over \( R_0^D \) and \( \mathcal{M}_n \) is a free patching \( \mathcal{R}_n^{\square,D} \)-module over \( M_0 \). Moreover, the surjections \( R_\infty \to R_n^{\square,D} \) from Theorem 2.5 induce an isomorphism \( R_\infty \cong \mathcal{P}(\mathcal{R}_n^{\square,D}) \).
Proof. Let $S'_n = \mathcal{O}[[y_1, \ldots, y_r]] \subseteq S_\infty$ with ideal $n' = (y_1, \ldots, y_r) = n \cap S'_n$, so that $R_n$ is a $S'_n$-algebra and $M_n$ is a $S'_n$-module. By definition, we have $R^n_\infty = R_n \otimes_{S'_\infty} S_\infty$ and $M^n_\infty = M_n \otimes_{S'_\infty} S_\infty$. Thus by Proposition 4.6,

$$\text{rank}_{S_\infty} R^n_\infty = \text{rank}_{S'_\infty} R_n = \text{rank}_\mathcal{O} R_0$$

$$\text{rank}_{S_\infty} M^n_\infty = \text{rank}_{S'_\infty} M_n = \text{rank}_\mathcal{O} M_0.$$

Also as $R^n_\infty$ is a quotient of $R^n_\infty$, we get that $\text{rank}_{S_\infty} R^n_\infty \leq \text{rank}_{S_\infty} R^n_\infty = \text{rank}_\mathcal{O} R_0$. Thus the $S_\infty$-ranks of the $R^n_\infty$'s and $M^n_\infty$'s are bounded so $\mathcal{R}^{\square, D}$ is a weak patching algebra and $\mathcal{M}^{\square}$ is a weak patching $\mathcal{R}$-module.

Also as noted above $R^n_\infty / n \cong R^D_0$ and $M^n_\infty / n \cong M_0$, so $\mathcal{R}^{\square, D}$ and $\mathcal{M}^{\square}$ are over $R^D_0$ and $M_0$, respectively.

Now by Proposition 4.7, for any $n$ we have,

$$I_n := \text{Ann}_{S_\infty} M^n_\infty = \text{Ann}_{S_\infty} \Lambda^n = \left( \text{Ann}_{S_\infty} M_\infty \right) = \left( (1 + y_1)^{e(n,1)}, \ldots, (1 + y_r)^{e(n,r)} \right) \subseteq S_\infty$$

(wheres as above, $e(n,i) \geq n$ for each $i$) and $M^n_\infty$ is free over $S_\infty / \text{Ann}_{S_\infty} M^n_\infty = \Lambda^n$.

It remains to show that $\mathcal{M}^{\square}$ is a patching system, i.e. that for any open $a \subseteq S_\infty$, $I_n = \text{Ann}_{S_\infty} M^n_\infty \subseteq a$ for all but finitely many $n$. But as $S_\infty / a$ is finite, and the group $1 + m_{S_\infty}$ is pro-$\ell$, the group $(1 + m_{S_\infty}) / a := \text{im}(1 + m_{S_\infty} \hookrightarrow S_\infty \to S_\infty / a)$ is a finite $\ell$-group. Since $1 + y_i \in 1 + m_{S_\infty}$ for all $i$, there is an integer $K \geq 0$ such that $(1 + y_i)^{\ell K} \equiv 1 \pmod{a}$ for all $i = 1, \ldots, d'$. Then for any $n \geq K$, $e(n,i) \geq n \geq K$ for all $i$, and so indeed $I_n \subseteq a$ by definition.

The final statement follows from Lemma 4.3 after noting that $R_\infty$ is a domain (by Theorem 3.2 and the discussion following it) and $\dim R_\infty = \dim S_\infty$.

Thus we may define $M_\infty := \mathcal{P}(\mathcal{M}^{\square})$. By Theorem 4.2 and Lemma 4.8 we get that $M_\infty$ is maximal Cohen–Macaulay over $\mathcal{P}(\mathcal{R}^{\square}) \cong R_\infty$ and

$$\dim_\mathcal{F} M_\infty / m_{R_\infty} = \dim_\mathcal{F} (M_\infty / n) / m_{R_\infty} = \dim_\mathcal{F} M_0 / m_{R_0} = \nu_\mathcal{F}(K^{\text{min}}).$$

### 4.4 The Properties of $M_\infty$

We shall now show that $M_\infty$ satisfies the remaining conditions of Theorem 3.1. We start by showing $\text{g. rank}_{R_\infty} M_\infty = 1$.

First, the fact that $R_\infty[1/\lambda]$ is formally smooth implies that:

**Lemma 4.9.** $M_0[1/\lambda]$ is free of rank 1 over $R^D_0[1/\lambda]$. In particular, the natural map $R^D_0[1/\lambda] \to \mathbb{T}_0[1/\lambda]$ is an isomorphism.

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Proof. As $M_\infty$ is maximal Cohen–Macaulay over $R_\infty$, $M_\infty[1/\lambda]$ is also maximal Cohen–Macaulay over $R_\infty[1/\lambda]$. Since $R_\infty[1/\lambda]$ is a formally smooth domain, this implies that $M_\infty[1/\lambda]$ is locally free over $R_\infty[1/\lambda]$ of some rank, $d$.

Now quotienting by $n$ we get that $M_\infty[1/\lambda]/n \cong M_0[1/\lambda]$ is locally free over $R_\infty[1/\lambda]/n \cong R^D_0[1/\lambda]$ of constant rank $d$. But now $R^D_0[1/\lambda]$ is a finite dimensional commutative $E$-algebra, and hence is a product of local rings. Thus as $M_0[1/\lambda]$ is locally free of rank $d$, it must actually be free of rank $d$.

But now by classical generic strong multiplicity 1 results we get that $M_0[1/\lambda]$ is free of rank 1 over $\mathbb{T}_0[1/\lambda]$ (recalling that $\mathbb{T}_0 = \mathbb{T}_\psi(K_0) \cong \mathbb{T}_\psi(K_{\min})$ and $M_0 = M_\psi(K_0) \cong M_\psi(K_{\min})$), which is a quotient of $R^D_0[1/\lambda]$. Thus $d = 1$ and hence $M_0[1/\lambda] \cong R^D_0[1/\lambda]$.

Lastly, as the action of $R^D_0[1/\lambda]$ on $M_0[1/\lambda]$ is free and factors through $R^D_0[1/\lambda] \twoheadrightarrow \mathbb{T}_0[1/\lambda]$, we get that $R^D_0[1/\lambda] \twoheadrightarrow \mathbb{T}_0[1/\lambda]$ is an isomorphism. 

It is now straightforward to compute $\text{g. rank}_{R_\infty} M_\infty$.

Let $K(R_\infty)$ and $K(S_\infty)$ be the fraction fields of $R_\infty$ and $S_\infty$, respectively. As $R_\infty$ is a finite type free $S_\infty$-algebra, $K(R_\infty)$ is a finite extension of $K(S_\infty)$. It follows that

$$M_\infty \otimes_{R_\infty} K(R_\infty) \cong M_\infty \otimes_{S_\infty} K(S_\infty).$$

Since $R_\infty$ and $M_\infty$ are both finite free $S_\infty$-modules, we thus get

$$\text{g. rank}_{R_\infty} M_\infty = \dim_{K(R_\infty)} [M_\infty \otimes_{R_\infty} K(R_\infty)] = \dim_{K(S_\infty)} [M_\infty \otimes_{S_\infty} K(S_\infty)]$$

$$= \frac{\dim_{K(S_\infty)} [M_\infty \otimes_{S_\infty} K(S_\infty)]}{\dim_{K(S_\infty)} K(R_\infty)} = \frac{\text{rank}_{S_\infty} M_\infty}{\text{rank}_{S_\infty} R_\infty}.$$

But now for any finite free $S_\infty$ module $A$ we have

$$\text{rank}_{S_\infty} A = \text{rank}_{S_\infty/n} A/n = \text{rank}_E A/n = \dim_E(A/n)[1/\lambda]$$

and so the fact that $M_\infty[1/\lambda]/n \cong M_0[1/\lambda] \cong R^D_0[1/\lambda] \cong R_\infty[1/\lambda]/n$ implies that $\text{rank}_{S_\infty} M_\infty = \text{rank}_{S_\infty} R_\infty$, giving $\text{g. rank}_{R_\infty} M_\infty = 1$.

Remark. It is worth mentioning here that Shotton’s computations of local deformation rings [Sho16] (particularly the fact that $R_\infty$ is Cohen–Macaulay, by Theorem 3.2) actually imply an integral “$R = T$” theorem. This result is likely known to experts, but we include it for the sake of completeness.

Specifically one considers the surjection $f : R^D_0 \twoheadrightarrow \mathbb{T}_0$. As shown in Lemma 4.9 (see also, [Kis09b]), $f$ is an isomorphism after inverting $\lambda$ (i.e. $R^D_0[1/\lambda] \cong \mathbb{T}_0[1/\lambda]$). This means that $\ker f \subseteq R^D_0$ is a torsion $O$-module.

But now $R_\infty$ is Cohen–Macaulay, and $M_\infty$ is a maximal Cohen–Macaulay module over $R_\infty$. Since $(\lambda, y_1, \ldots, y_r, w_1, \ldots, w_j)$ is a regular sequence for $M_\infty$ (by Theorem 4.2(3)) it follows that it is also

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a regular sequence for \( R_\infty \). Thus \( R_0^D \cong R_\infty / n = R_\infty / (y_1, \ldots, y_r, w_1, \ldots, w_j) \) is Cohen–Macaulay and \( \lambda \) is a regular element on \( R_0^D \) (i.e. a non zero divisor).

But this implies that \( R_0^D \) is \( \lambda \)-torsion free, giving that \( \ker f = 0 \), so indeed \( f : R_0^D \to \mathbb{T}_0 \) is an isomorphism.

It remains to show the following:

**Proposition 4.10.** \( M_\infty \cong M_\infty^* := \text{Hom}_{R_\infty}(M_\infty, \omega_{R_\infty}) \) as \( R_\infty \)-modules.

This will ultimately follow from the fact that the modules \( M(K) \) were naturally self-dual:

**Lemma 4.11.** For any \( n \geq 1 \), there is a \( \mathbb{T}^D(K_n)_m \)-equivariant perfect pairing \( M(K_n) \times M(K_n) \to \mathcal{O} \). This induces a \( \mathbb{T}^D(K_n)_m \)-equivariant perfect pairing \( M_\psi(K_n) \times M_\psi(K_n) \to \mathcal{O} \), and thus a \( \mathbb{T}_n \)-equivariant perfect pairing \( M_n \times M_n \to \mathcal{O} \).

**Proof.** First note that there is a \( \mathbb{T}^D(K_n) \)-equivariant perfect pairing \( S^D(K_n) \times S^D(K_n) \to \mathcal{O} \). In the totally definite case, this is the monodromy pairing, in the indefinite case it is Poincaré duality (although this must be modified slightly in order to make the pairing \( \mathbb{T}^D(K_n) \)-equivariant, see [Car94, 3.1.4]). Completing and dualizing gives a \( \mathbb{T}^D(K_n)_m \)-equivariant perfect pairing \( S^D(K_n)_m^\vee \times S^D(K_n)_m^\vee \to \mathcal{O} \).

In the totally definite case, this is already the desired pairing \( M(K_n) \times M(K_n) \to \mathcal{O} \). In the indefinite case, it follows from [Car94, 3.2.3] that the self-duality on \( S^D(K_n)_m^\vee \) implies that \( M(K_n) = \text{Hom}_{R_n[G_F]}(\rho_{\text{univ}}, S^D(K_n)_m^\vee) \) is also \( \mathbb{T}^D(K_n)_m \)-equivariantly self-dual.

Now in the notation above we have \( \mathcal{O}[C_K] = \mathcal{O}[S_v]_{v \not\in S} \subseteq \mathbb{T}^D(K_n)_m \subseteq \mathbb{T}^D(K_n)_m, M(K_n) \) is a finite free \( \mathcal{O}[C_K] \)-module and \( M_\psi(K_n) = M(K_n)/\mathfrak{J}_\psi M(K_n) \). To deduce the pairing on \( M_\psi(K_n) \), it will suffice to show that \( M(K)/\mathfrak{J}_\psi M(K) \cong M(K)/[\mathfrak{J}_\psi] \) as \( \mathbb{T}^D(K_n)_m \)-modules.

Consider the character \( \phi : C_K \to \mathcal{O}^\times \) defined \( \phi(\overline{w_v}) = \psi(\text{Frob}_v)/\text{Nm}(v) \) for all \( v \not\in S \) (this is well defined by the construction of \( C_K \)), and note that \( \mathfrak{J}_\psi \) is generated by the elements \( g - \phi(g) \) for all \( g \in C_K \). Let \( a_\psi = \sum_{g \in C_K} \phi(g)^{-1} g \in \mathcal{O}[C_K] \subseteq \mathbb{T}^D(K)_m \). The standard theory of group rings implies that multiplication by \( a_\psi \) induces a short exact sequence

\[
0 \to \mathfrak{J}_\psi \to \mathcal{O}[C_K] \xrightarrow{a_\psi} (\mathcal{O}[C_K])[\mathfrak{J}_\psi] \to 0
\]

Since \( M(K) \) is a free \( \mathcal{O}[C_K] \)-module, this implies the multiplication by \( a_\psi \) induces an isomorphism \( a_\psi : M(K)/\mathfrak{J}_\psi M(K) \cong M(K)/[\mathfrak{J}_\psi] \). As \( a_\psi \in \mathcal{O}[C_K] \subseteq \mathbb{T}^D(K)_m \), this is the desired isomorphism of \( \mathbb{T}^D(K)_m \)-modules, and so we indeed have a \( \mathbb{T}^D(K)_m \)-equivariant perfect pairing \( M_\psi(K) \times M_\psi(K) \to \mathcal{O} \).

The final statement, about the pairing \( M_n \times M_n \to \mathcal{O} \) follows by localizing at \( mQ_n \).  

To deduce Proposition 4.10 from Lemma 4.11, we shall make use of the following lemma:

**Lemma 4.12.** If $A$ is a local Cohen–Macaulay ring and $B$ is a local $A$-algebra which is also Cohen–Macaulay with $\dim A = \dim B$, then for any $B$-module $M$,

$\text{Hom}_A(M, \omega_A) \cong \text{Hom}_B(M, \omega_B)$

as left $\text{End}_B(M)$-modules.

**Proof.** By [Sta18, Tag 08YP] there is an isomorphism

$\text{Hom}_A(M, \omega_A) \cong \text{Hom}_B(M, \text{Hom}_A(B, \omega_A))$

sending $\alpha : M \to \omega_A$ to $\alpha' : m \mapsto (b \mapsto \alpha(bm))$, which clearly preserves the action of $\text{End}_B(M)$ (as $(\alpha \circ \psi)(bm) = \alpha(b\psi(m))$ for any $\psi \in \text{End}_B(M)$). It remains to show that $\text{Hom}_A(B, \omega_A) \cong \omega_B$, which is just Theorem 21.15 from [Eis95] in the case $\dim A = \dim B$.

**Proof of Proposition 4.10.** By Lemma 4.11, we have $M_n \cong \text{Hom}_{\mathcal{O}}(M_n, \mathcal{O})$ as $R^n_\mathcal{O}$-modules for all $n \geq 1$. Now as $\Delta_n$ is a finite group, the ring $\Lambda_n = \mathcal{O}[\Delta_n]$ has Krull dimension 1. Moreover as in the proof of Lemma 4.8, $\Lambda_n = \mathcal{O}.[[y_1, \ldots, y_r]]/I_n$, where $I_n$ is generated by $r$ elements. Thus $\Lambda_n$ is a complete intersection, and so $\omega_{\Lambda_n} = \Lambda_n$. Thus by Lemma 4.12 we have $M_n \cong \text{Hom}_{\Lambda_n}(M_n, \Lambda_n)$, again as $R^n_\mathcal{O}$-modules.

Tensoring with $\mathcal{O}.[[w_1, \ldots, w_j]]$, this implies $M^n_\mathcal{O} \cong \text{Hom}_{\Lambda^n_\mathcal{O}}(M^n_\mathcal{O}, \Lambda^n_\mathcal{O})$ as $R^n_\mathcal{O}$-modules (and hence as $R_\mathcal{O}$-modules). Moreover, by Lemma 4.6, $M^n_\mathcal{O}$ is finite free over $\Lambda^n_\mathcal{O}$.

Now take any open ideal $\mathfrak{a} \subseteq S_\infty$. Letting $\Lambda^n_\mathfrak{a} = S_\infty/I_n$ as in the proof of Lemma 4.8 we have that $I_n \subseteq \mathfrak{a}$ for all but finitely many $n$. For any such $n$, we now have:

$$M^n_\mathcal{O}/\mathfrak{a} \cong \text{Hom}_{\Lambda^n_\mathfrak{a}}(M^n_\mathcal{O}, \Lambda^n_\mathfrak{a})/\mathfrak{a} \cong \text{Hom}_{\Lambda^n_\mathcal{O}}(M^n_\mathcal{O}, \Lambda^n_\mathcal{O}/\mathfrak{a}) = \text{Hom}_{\Lambda^n_\mathfrak{a}}(M^n_\mathcal{O}, S_\infty/\mathfrak{a})$$

$$= \text{Hom}_{S_\infty/\mathfrak{a}}(M^n_\mathcal{O}/\mathfrak{a}, S_\infty/\mathfrak{a})$$

as $R^n_\mathfrak{a}$-modules.

Now as noted above, we have that $\mathcal{U}(\mathcal{R}_i^n/\mathfrak{a}) \cong R^n_\mathfrak{a}$ and $\mathcal{U}(\mathcal{M}_i^n/\mathfrak{a}) \cong M^n_i/\mathfrak{a}$ for $\mathfrak{a}$-many $i$. Taking any such $i$, the above computation gives that:

$$\mathcal{U}(\mathcal{M}_i^n/\mathfrak{a}) \cong \text{Hom}_{S_\infty/\mathfrak{a}}(\mathcal{U}(\mathcal{M}_i^n/\mathfrak{a}), S_\infty/\mathfrak{a})$$

as $\mathcal{U}(\mathcal{R}_i^n/\mathfrak{a})$-modules. Taking inverse limits, it now follows that:

$$M_\infty = \mathcal{P}(\mathcal{M}_i^n) \cong \varprojlim \text{Hom}_{S_\infty/\mathfrak{a}}(\mathcal{U}(\mathcal{M}_i^n/\mathfrak{a}), S_\infty/\mathfrak{a})$$

as $\mathcal{P}(\mathcal{R}_i^n)$-modules. Now we claim that the right hand side is just $\text{Hom}_{S_\infty}(M_\infty, S_\infty)$. Using the fact that $M_\infty$, and thus $\text{Hom}_{S_\infty}(M_\infty, S_\infty)$, is a finite free $S_\infty$-module (and thus is $m_{S_\infty}$-adically complete) we get that:

$$\text{Hom}_{S_\infty}(M_\infty, S_\infty) \cong \varprojlim \text{Hom}_{S_\infty}(M_\infty, S_\infty)/\mathfrak{a}$$
as $\mathcal{P}(\Box, \mathcal{D}) = \lim_{\to} a \mathcal{P}(\Box, \mathcal{D})/a$-modules. But now for any $a$, as $M_\infty \cong \mathcal{P}(\mathcal{M})$ is a projective $S_\infty$-module:

$$\text{Hom}_{S_\infty}(M_\infty, S_\infty)/a \cong \text{Hom}_{S_\infty/a}(M_\infty/a, S_\infty/a) \cong \text{Hom}_{S_\infty/a}(U(\mathcal{M}/a), S_\infty/a)$$

as $\mathcal{P}(\Box, \mathcal{D})/a = U(\mathcal{P}(\Box, \mathcal{D})/a)$-modules. So indeed:

$$\text{Hom}_{S_\infty}(M_\infty, S_\infty) \cong \lim_{\to} a \text{Hom}_{S_\infty/a}(U(\mathcal{M}/a), S_\infty/a) \cong M_\infty$$

as $\mathcal{P}(\Box, \mathcal{D})$-modules, and hence as $R_\infty$-modules.

But now as $\dim R_\infty = \dim S_\infty$ and $S_\infty$ is regular (and thus Gorenstein), Lemma 4.12 implies that

$$M_\infty \cong \text{Hom}_{S_\infty}(M_\infty, S_\infty) \cong \text{Hom}_{R_\infty}(M_\infty, \omega_{R_\infty})$$

as $R_\infty$-modules, as claimed. □

This shows that $M_\infty$ indeed satisfies the conditions of Theorem 3.1, and so completes the proof of Theorem 1.1.

4.5 Endomorphisms of Hecke modules

It remains to show Theorem 1.2. We first note that Theorem 1.2 can be restated in terms of the objects considered the previous section as follows:

**Proposition 4.13.** The trace map $M(K_0) \otimes_{T D(K_0)m} M(K_0) \to \omega_{T D(K_0)m}$ induced by the perfect pairing from Lemma 4.11 is surjective, and the natural map $T D(K_0)m \to \text{End}_{T D(K_0)m}(M(K_0))$ is an isomorphism.

**Proof that Proposition 4.13 implies Theorem 1.2.** First note that by Lemma 4.5 it suffices to prove Theorem 1.2 with all of the levels $K_{\text{min}}$ replaced by $K_0$. If $D$ is definite, then self-duality and the definition of $M(K_0)$ implies that $M(K_0) \cong S^D(K_0)^\vee_m \cong S^D(K_0)m$, and so the statement of Theorem 1.2 is immediate.

Now assume that $D$ is indefinite. To simply notation, write $T = T D(K_0)m$, $M = M(K_0)$, $S_{K_0} = S^D(K_0)m$ and $R = R_{F,S}$. As noted above, we have an isomorphism

$$M \otimes_R \rho^{\text{univ}} = \text{Hom}_{R[G_F]}(\rho^{\text{univ}}, S_{K_0}^\vee) \otimes_R \rho^{\text{univ}} \cong S_{K_0}^\vee$$
by [Car94]. It now follows that

\[
\text{End}_{R[G_F]}(S_{K_0}) \cong \text{End}_{R[G_F]}(S^\vee_{K_0}) \\
\cong \text{Hom}_{R[G_F]}(M \otimes_R \rho^{\text{univ}}, S^\vee_{K_0}) \\
\cong \text{Hom}_{R[G_F]}((R[G_F] \otimes R[G_F]) \otimes \rho^{\text{univ}}, S^\vee_{K_0}) \\
\cong \text{Hom}_{R[G_F]}(M \otimes_R R[G_F], \text{Hom}_{R[G_F]}(\rho^{\text{univ}}, S^\vee_{K_0})) \\
\cong \text{Hom}_R(M, \text{Hom}_{R[G_F]}(\rho^{\text{univ}}, S^\vee_{K_0})) \\
\cong \text{Hom}_R(M, M) = \text{End}_R(M) = \text{End}_T(M) = T
\]

as $T$-algebras (where we used [Sta18, Tag 00DE, Tag 05DQ]).

The above work implies the following “fixed-determinant” version of Proposition 4.13:

**Proposition 4.14.** The trace map $M_\psi(K_0) \otimes_{T_\psi(K_0)_m} M_\psi(K_0) \to \omega_{T_\psi(K_0)_m}$ induced by the perfect pairing from Lemma 4.11 is surjective, and the natural map $T_\psi(K_0)_m \to \text{End}_{T_\psi(K_0)_m}(M_\psi(K_0))$ is an isomorphism. Moreover, the surjection $R_{F,S}^{D,\psi}(\mathfrak{p}) \to T_\psi(K_0)_m$ from Lemma 2.4 is an isomorphism.

**Proof.** Recall that in the notation of Section 4.3, $M_\psi(K_0) = M_0$, $T_\psi(K_0)_m = T_0$ and $R_{F,S}^{D,\psi}(\mathfrak{p}) = R_0^D$. As shown above, $M_\infty$ satisfies the hypotheses of Theorem 3.1, so by the last conclusion of Theorem 3.1, we get that the trace map $\tau_{M_\infty} : M_\infty \otimes_{R_\infty} M_\infty \to \omega_{R_\infty}$ is surjective.

As noted above, $(y_1, \ldots, y_r, w_1, \ldots, w_j)$ is a regular sequence for $M_\infty$, and hence for $R_\infty$. It follows that $R_0^D \cong R_\infty / \mathfrak{n}$ is Cohen–Macaulay and $M_0 \cong M_\infty / \mathfrak{n}$ is maximal Cohen–Macaulay over $R_0^D$. Moreover, we get that the dualizing module of $R_0^D$ is just $\omega_{R_0^D} \cong \omega_{R_\infty} / \mathfrak{n}$.

But now quotienting out by $\mathfrak{n}$, we thus get a surjective map $M_0 \otimes_{R_0^D} M_0 \to \omega_{R_0^D}$, which (by Lemma 3.4) implies that the trace map $\tau_{M_0} : M_0 \otimes_{R_0^D} M_0 \to \omega_{R_0^D}$ is also surjective.

But now, as in [Eme02, Lemmas 2.4 and 2.6], we have the following commutative algebra result:

**Lemma 4.15.** Let $B$ be an $O$-algebra and let $U$ and $V$ be $B$-modules. Assume that $B, U$ and $V$ are all finite rank free $O$-modules, and we have a $B$-bilinear perfect pairing $\langle \, , \, \rangle : V \times U \to O$. Moreover, assume that $U[1/\lambda]$ is free over $B[1/\lambda]$. Define $\phi : U \otimes_B V \to \text{Hom}_O(B, O)$ by $\phi(u \otimes v)(b) = \langle bu, v \rangle = \langle u, bv \rangle$. Then $\phi$ is surjective if and only if the natural map from $B$ to the center of $\text{End}_B(U)$ is an isomorphism.

Applying this with $B = R_0^D$, $U = M_0$, $V = M_0^*$ and $\langle \, , \, \rangle : M_0 \times M_0 \to O$ being the perfect pairing from Lemma 4.11 implies that the natural map $R_0^D \to Z(\text{End}_{R_0^D}(M_0))$ is an isomorphism. (Here we have used the fact that $\omega_{R_0^D} \cong \text{Hom}_O(R_0^D, O)$ as in the proof of Lemma 4.12, and $M_0[1/\lambda] \cong R_0^D[1/\lambda]$ by Lemma 4.9.)
But now as $M_0$ is free over $\mathcal{O}$, we get that

$$\text{End}_{R^D_0}(M_0) \hookrightarrow \text{End}_{R^D_0[1/\lambda]}(M_0[1/\lambda]) \cong \text{End}_{R^D_0[1/\lambda]}(R^D_0[1/\lambda]) = R^D_0[1/\lambda]$$

and so $\text{End}_{R^D_0}(M_0)$ is commutative. Hence the natural map $R^D_0 \to \text{End}_{R^D_0}(M_0)$ is an isomorphism. As the action of $R^D_0$ on $M_0$ factors through $R^D_0 \to T_0$, this implies that this map and the map $T_0 \to \text{End}_{T_0}(M_0)$ are isomorphisms.$^9$

It remains to deduce Proposition 4.13 from Proposition 4.14. As in the proof of Proposition 4.13, write $T := T^D(K_0)_m$, $M := M(K_0)$ and $R := R_{F,S}(\overline{\mathfrak{p}})$. Also write $T_\psi := T^D_0(K_0)_m$, $M_\psi := M_\psi(K_0)$ and $T_\psi := R_{F,S}(\overline{\mathfrak{p}})$ and $T^D_\psi = R_{F,S}(\overline{\mathfrak{p}}) \cong T_\psi$. Lastly recall the group $C_{K_0,\ell}$ from Section 4.2 above, and abbreviate this as $C := C_{K_0,\ell}$. Recall that $\mathcal{O}[C]$ is a subalgebra of $T$ and $M$ is free over $\mathcal{O}[C]$ with we have $M_\psi = M/\mathcal{I}_\psi$.

Our argument will hinge on the following key fact about the structure of $T$:

**Lemma 4.16.** There is an isomorphism $T \cong T_\psi \otimes_\mathcal{O} \mathcal{O}[C]$.

**Proof.** As in Lemma 4.9 we have $T_\psi[1/\lambda] \cong M_\psi[1/\lambda]$ by classical generic multiplicity 1 results. An analogous argument gives $T[1/\lambda] \cong M[1/\lambda]$. As $T, T_\psi, M$ and $M_\psi$ are all finite rank free $\mathcal{O}$-modules, this gives $\text{rank}_\mathcal{O} T_\psi = \text{rank}_\mathcal{O} M_\psi$ and $\text{rank}_\mathcal{O} T = \text{rank}_\mathcal{O} M$. Also as $M$ is free over $\mathcal{O}[C]$, $M/\mathcal{I}_\psi \cong M_\psi$ and $\mathcal{O}[C]/\mathcal{I}_\psi \cong \mathcal{O}$, we get that $\text{rank}_\mathcal{O} M = (\text{rank}_\mathcal{O} M_\psi)(\mathcal{O}[C])$. It follows that $\text{rank}_\mathcal{O} T = \text{rank}_\mathcal{O}(T_\psi \otimes_\mathcal{O} \mathcal{O}[C])$. As all of these rings are free over $\mathcal{O}$, it will thus suffice to give a surjection $T_\psi \otimes_\mathcal{O} \mathcal{O}[C] \twoheadrightarrow T$.

Now recall the isomorphism $R_{F,S}(\overline{\psi}) \otimes_\mathcal{O} R_\psi \simeq R$ from Lemma 2.3. This gives us maps $\alpha : R_{F,S}(\overline{\psi}) \to R \to T$ and $\beta : R_\psi \to R \to T$ such that $(\text{im} \alpha)(\text{im} \beta) = T$. It will thus suffice to show that $\alpha$ and $\beta$ factor through surjections $R_{F,S}(\overline{\psi}) \to \mathcal{O}[C]$ and $R_\psi \to T_\psi$, respectively.

Now note that the map $R \to T$ from Lemma 2.4 is induced by a representation $\rho^\text{mod} : G_{F,S} \to \text{GL}_2(\mathbb{T})$ lifting $\overline{\mathfrak{p}}$, and we have $\text{tr}(\rho^\text{mod}(\text{Frob}_v)) = T_v$ and $\det(\rho^\text{mod}(\text{Frob}_v)) = \text{Nm}(v)S_v$ for all $v \not\in S$ (cf [Kis09b]).

By Lemma 2.3, the map $R_{F,S}(\overline{\psi}) \to R$ is characterized by the morphism of functors $\rho \mapsto \det \rho$, and so satisfies $\psi^\text{univ}(g) \mapsto \det \rho^\text{univ}(g)$ for all $g \in G_{F,S}$ (where $\psi^\text{univ}$ is the universal lift of $\overline{\psi}$). Thus for any $v \not\in S$, we have $\alpha(\psi^\text{univ}(\text{Frob}_v)) = \det \rho^\text{mod}(\text{Frob}_v) = \text{Nm}(v)S_v$. By Chebotarev density, it follows that $\text{im} \alpha = \mathcal{O}[\mathcal{S}_v]_{v \not\in S} = \mathcal{O}[C] \subseteq T$.

Now for any map $x : T \to \overline{\mathbb{E}}$, let $\rho_x : G_{F,S} \to \text{GL}_2(\mathbb{T}) \xrightarrow{x} \text{GL}_2(\overline{\mathbb{E}})$ be the modular lift of $\overline{\mathfrak{p}}$ corresponding to $x$. Let $\rho^\psi_{x \circ \beta} : G_{F,S} \to \text{GL}_2(R_\psi) \xrightarrow{\rho_{x \circ \beta}} \text{GL}_2(\overline{\mathbb{E}})$ be the lift of $\overline{\mathfrak{p}}$ with determinant $\psi$ corresponding to $x \circ \beta : R_\psi \to R \to T \to \overline{\mathbb{E}}$. From the construction of the map $R_\psi \to R$, we

$^9$As noted in the remark following Lemma 4.9, the fact that $R^D_0 \to T_0$ is an isomorphism follows more simply from the fact that $R_\infty$ is a Cohen-Macaulay domain, flat over $\mathcal{O}$ (as shown in [Sho16]) and the fact that $R_0[1/\ell] \cong T_0[1/\ell]$ (shown in [Kis09b]).
see that $\rho_{x,0}^\psi$ is the (unique) twist of $\rho_x$ with determinant $\psi$. In particular, $\rho_{x,0}^\psi$ is also modular of level $K_0$. It follows that $\rho_{x,0}^\psi$ is flat at every $v|\ell$, Steinberg at every $v\mid D$, and minimal level at every $v \in \Sigma_D$, $v \nmid \ell, D$. As remarked in Section 2.2, this implies that $x \circ \beta : R_\psi \to T \to \mathbb{E}$ factors through $R_{\psi}^D \cong T_\psi$ for any $x : T \to \mathbb{E}$.

But now as $T$ is a finite free $O$-module, we have an injection $T \hookrightarrow T \otimes_O \mathbb{E}$, and as $T$ is reduced, $T \otimes_O \mathbb{E}$ is a finite product of copies of $\mathbb{E}$. Thus by the above, the map $R_\psi \beta : T \hookrightarrow T \otimes_O \mathbb{E}$ factors through $R_\psi \to R_{\psi}^D \cong T_\psi$, and so $\beta : R_\psi \to T$ does as well. So indeed $\beta \otimes \alpha$ induces a surjection (and hence an isomorphism) $T_\psi \otimes O[C] \to T$.

Now as $O[C]$ is a complete intersection, we have $\omega_{O[C]} \cong O[C]$. Thus using Lemma 4.12 we get

$$\omega_T \cong \text{Hom}_{O[C]}(T, O[C]) \cong \text{Hom}_{O[C]}(T \otimes_O O[C], O[C]) \cong \text{Hom}_O(T_\psi, O) \otimes_O O[C] \cong \omega_{T_\psi} \otimes_O O[C].$$

In particular, $\omega_T / J_\psi \cong \omega_{T_\psi}$. Now consider the trace map $\tau_M : M \otimes_T M \to \omega_T$, and notice that the induced map $\tau_M \otimes_{O[C]} (O[C] / J_\psi) : (M / J_\psi) \otimes_T (M / J_\psi) \to \omega_T / J_\psi$ can be identified with $\tau_{M_\psi} : M_\psi \otimes_{T_\psi} M_\psi \to \omega_{T_\psi}$. Since $- \otimes_{O[C]} (O[C] / J_\psi)$ is right-exact and $\tau_{M_\psi}$ is surjective, it follows that $\tau_M$ is surjective as well.

It now follows by the argument in the proof of Proposition 4.14 that the map $T \to \text{End}_T(M)$ is an isomorphism. This completes the proof the Proposition 4.13, and hence of Theorem 1.2.

References


