In these notes we will develop the commutative algebra results needed for the Taylor–Wiles–Kisin patching method, as reformulated by Scholze in [Sch15].

1 Ultraproducts

Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the natural numbers. Recall that a nonprincipal ultrafilter on $\mathbb{N}$ is a collection, $\mathcal{F}$, of subsets of $\mathbb{N}$ satisfying the following conditions:

1. $\mathcal{F}$ does not contain any finite sets.
2. If $I, J \in \mathcal{F}$ then $I \cap J \in \mathcal{F}$
3. If $I \in \mathcal{F}$ and $I \subseteq J \subseteq \mathbb{N}$, then $J \in \mathcal{F}$ as well.
4. If $I \cup J = \mathbb{N}$ is a partition of $\mathbb{N}$, then either $I \in \mathcal{F}$ or $J \in \mathcal{F}$.

It is well known that such an $\mathcal{F}$ must exist, if one assumes the axiom of choice.

Note that these conditions imply the following: If $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n = \mathbb{N}$ is a partition of $\mathbb{N}$, then $I_i \in \mathcal{F}$ for exactly one $i$.

For the remainder of this appendix, we will fix a nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.

For convenience, we will say that a property $P(i)$ holds for $\mathcal{F}$-many $i$ if there is some $I \in \mathcal{F}$ such that $P(i)$ is true for all $i \in I$. The four conditions above imply the following:

1. If $P(i)$ holds for $\mathcal{F}$-many $i$, then it holds for infinitely many $i$.
2. If $P(i)$ and $Q(i)$ each hold for $\mathcal{F}$-many $i$, then $P(i)$ and $Q(i)$ are simultaneously true for $\mathcal{F}$-many $i$.
3. $P(i)$ holds for $\mathcal{F}$-many $i$ if and only if the set $\{i | P(i) \text{ is true}\}$ is in $\mathcal{F}$.
4. For any property $P$, either $P(i)$ is true for $\mathcal{F}$-many $i$, or it is false for $\mathcal{F}$-many $i$.

If $\mathcal{M} = \{M_n\}_{n \geq 1}$ is any sequence of sets, we define an equivalence relation $\sim$ on the set $\prod_{n \geq 1} M_n$ by $(m_1, m_2, \ldots) \sim (m'_1, m'_2, \ldots)$ if $m_i = m'_i$ for $\mathcal{F}$-many $i$ (the above properties of ultrafilters imply
that this is an equivalence relation). We then define the ultraproduct of \( \mathcal{M} \) to be

\[
\mathcal{U}(\mathcal{M}) := \left( \prod_{n \geq 1} M_n \right) / \sim
\]

For any \( m = (m_1, m_2, \ldots) \in \prod_{n \geq 1} M_n \) we will denote the equivalence class of \( m \) in \( \mathcal{U}(\mathcal{M}) \) by \( [m_i]_i = [m_1, m_2, \ldots] \). We will frequently define elements \( m = [m_i]_i \) by only specifying \( m_i \) for \( \mathcal{F} \)-many \( i \). Doing so is unambiguous, as if \( m_i \) is specified for all \( i \in I \) (\( I \in \mathcal{F} \)) the choices of \( m_j \) for \( j \in \mathbb{N} \setminus I \) do not affect the equivalence class \( [m_i]_i \).

If \( M \) is any set we will write \( M := \{M\}_{n \geq 1} \) for the constant sequence of sets, and define the ultrapower of \( M \) to be \( M^U := \mathcal{U}(M) \). Notice that we have a diagonal map \( \Delta : M \to M^U \) defined by \( m \mapsto [m, m, \ldots] \). This map is clearly injective.

In our applications, we will generally consider the case where each \( M_n \) has a certain algebraic structure. Thus for the rest of this subsection we will fix a category, \( C \) of sets with algebraic structure, taken to be one of the following:

- The category of abelian groups;
- The category of (commutative) rings;
- The category of (continuous) \( R \)-modules;
- The category of (continuous) \( R \)-algebras,

for some fixed ring topological \( R \) (which we will often take to have the discrete topology, however the continuous version will be used in Lemma 3.2). Using the language of universal algebra (or more generally, of model theory) it is possible phrase the results of this section for significantly more general categories of “sets with structure,” however the specific cases we discuss here will be sufficient for our purposes.

We first show that if each \( M_n \) is in \( C \), then \( \mathcal{U}(\mathcal{M}) \) inherits a natural \( C \)-object structure.

**Proposition 1.1.** Let \( \mathcal{M} = \{M_n\}_{n \geq 1} \), and assume that each \( M_n \) is in \( C \). Then \( \mathcal{U}(\mathcal{M}) \) may be given the structure of object in \( C \) with the operations additions, multiplication and scalar multiplication (when appropriate) defined by:

\[
\begin{align*}
[a_1, a_2, \ldots] + [b_1, b_2, \ldots] &= [a_1 + b_1, a_2 + b_2, \ldots] \\
[a_1, a_2, \ldots] \cdot [b_1, b_2, \ldots] &= [a_1 \cdot b_1, a_2 \cdot b_2, \ldots] \\
r[a_1, a_2, \ldots] &= [ra_1, ra_2, \ldots]
\end{align*}
\]

for \( \alpha = [a_1, a_2, \ldots], \beta = [b_1, b_2, \ldots] \in \mathcal{U}(\mathcal{M}) \), the elements \( 0, 1 \in \mathcal{U}(\mathcal{M}) \) (again when appropriate) defined by:

\[
0 = [0, 0, \ldots] \in \mathcal{U}(\mathcal{M}) , \quad 1 = [1, 1, \ldots] \in \mathcal{U}(\mathcal{M}) ,
\]

and topology defined by the quotient map \( \pi : \prod_{n \geq 1} M_n \to \mathcal{U}(\mathcal{M}) \). Moreover:
1. The natural surjection \( \pi : \prod_{n \geq 1} M_n \to \mathcal{U}(\mathcal{M}) \), \((m_i)_i \mapsto [m_i]_i\) is a \(C\)-morphism.

2. For \(M \in C\), the diagonal map \(\Delta : M \to M^d\) is a \(C\)-morphism.

**Proof.** We will prove this only in the case when \(C\) is taken to by the category of continuous \(R\)-algebras. The other cases are analogous.

First we check that the operations are well-defined. Take \(\alpha = [a_i], \alpha' = [a'_i], \beta = [b_i], \beta' = [b'_i] \in \mathcal{U}(\mathcal{M})\) with \(\alpha = \alpha'\) and \(\beta = \beta'\). Then for \(\mathfrak{F}\)-many \(i\) we simultaneously have that \(a_i = a'_i\) and \(b_i = b'_i\). It follows that \(a_i + b_i = a'_i + b'_i\), \(a_i \cdot b_i = a'_i \cdot b'_i\) and \(r a_i = r a'_i\) for \(\mathfrak{F}\)-many \(i\), and so \(\alpha + \beta = \alpha' + \beta'\), \(\alpha \cdot \beta = \alpha' \cdot \beta'\) and \(r \alpha = r \alpha'\).

Now as the operations are defined pointwise, they are clearly preserved by \(\pi : \prod_{n \geq 1} M_n \to \mathcal{U}(\mathcal{M})\). Thus as \(\prod_{n \geq 1} M_n\) is a continuous \(R\)-algebra, and \(\pi\) is continuous by definition, (1) will follow if we show that the operations make \(\mathcal{U}(\mathcal{M})\) into a \(R\)-algebra (the operations will automatically be continuous as \(\mathcal{U}(\mathcal{M})\) has the quotient topology).

Now let

\[
K = \left\{ (a_1, a_2, \ldots) \in \prod_{n \geq 1} M_n \mid (a_1, a_2, \ldots) \sim (0, 0, \ldots) \right\} 
\]

\[
= \left\{ (a_1, a_2, \ldots) \in \prod_{n \geq 1} M_n \mid a_i = 0 \text{ for } \mathfrak{F}\text{-many } i \right\} \subseteq \prod_{n \geq 1} M_n
\]

Now as the operations are well-defined, for any \(a = (a_n)_n, b = (b_n)_n \in K\), any \(m = (m_n)_n \in \prod_{n \geq 1} M_n\) and any \(r \in R\) we get that:

\[
(a_n + b_n)_n = (a_n)_n + (b_n)_n \sim (0)_n + (0)_n = (0)_n \\
(m_n \cdot a_n)_n = (m_n)_n \cdot (a_n)_n \sim (m_n)_n \cdot (0)_n = (0)_n \\
r(a_n)_n = r(a_n)_n \sim r(0)_n = (0)_n,
\]

and so \(a + b, ma, ra \in K\). It follows that \(K \subseteq \prod_{n \geq 1} M_n\) is an ideal.

Also by definition, for \(a = (a_n)_n, b = (b_n)_n \in \prod_{n \geq 1} M_n\), \(a \sim b\) if and only if \(a - b \in K\). It follows that \(\pi : \prod_{n \geq 1} M_n \to \mathcal{U}(\mathcal{M})\) gives an identification \(\overline{\pi} : \left( \prod_{n \geq 1} M_n \right) / K \to \mathcal{U}(\mathcal{M})\). As \(\pi\), and thus \(\overline{\pi}\), preserves the operations and \(\left( \prod_{n \geq 1} M_n \right) / K\) is an \(R\)-algebra, it follows that \(\mathcal{U}(\mathcal{M})\) is indeed an \(R\)-algebra, and \(\pi\) is an \(R\)-algebra homomorphism. This proves (1).
Given two sequences \( \mathcal{M} = \{M_n\}_{n \geq 1} \) and \( \mathcal{M}' = \{M'_n\}_{n \geq 1} \) in \( \mathcal{C} \), we define an \( \mathfrak{F} \)-morphism \( \varphi : \mathcal{M} \to \mathcal{M}' \) to be a collection of \( \mathfrak{F} \)-morphisms \( \varphi = \{\varphi_i : M_i \to M'_i\}_{i \in I} \) indexed by some \( I \in \mathfrak{F} \). Then we have

**Proposition 1.2.** If \( \varphi : \mathcal{M} \to \mathcal{M}' \) is an \( \mathfrak{F} \)-morphism, then the map \( \varphi^M : \mathcal{U}(\mathcal{M}) \to \mathcal{U}(\mathcal{M}') \) given by \( \varphi^M[a_i]_i = [\varphi_i(a_i)]_i \) is a well-defined \( \mathcal{C} \)-morphism. Moreover,

1. If \( \varphi, \psi : \mathcal{M} \to \mathcal{M}' \) are two \( \mathfrak{F} \)-morphisms, and \( \varphi_i = \psi_i \) for \( \mathfrak{F} \)-many \( i \), then \( \varphi^M = \psi^M \). In particular, if \( \varphi : \mathcal{M} \to \mathcal{M} \) satisfies \( \varphi_i = \id_{M_i} : M_i \to M_i \) for \( \mathfrak{F} \)-many \( i \), then \( \varphi^M = \id_{\mathcal{U}(\mathcal{M})} : \mathcal{U}(\mathcal{M}) \to \mathcal{U}(\mathcal{M}) \).

2. For two \( \mathfrak{F} \)-morphisms, \( \varphi : \mathcal{M} \to \mathcal{M}' \) and \( \psi : \mathcal{M}' \to \mathcal{M}'' \), we have \( \psi^M \circ \varphi^M = (\psi \circ \varphi)^M \).

Hence \( \mathcal{U}(\cdot) \) is a functor.

**Proof.** As in Proposition 1.1, we will prove this only in the case where \( \mathcal{C} \) is the category of continuous \( R \)-algebras.

Let \( \varphi : \mathcal{M} \to \mathcal{M}' \) be an \( \mathfrak{F} \)-morphism. If we have \( [a_i]_i = [a'_i]_i \) in \( \mathcal{U}(\mathcal{M}) \), then for \( \mathfrak{F} \)-many \( i \) we simultaneously have that \( \varphi_i \) exists and \( a_i = a'_i \). Thus \( \varphi^M[a_i]_i = [\varphi_i(a_i)]_i = \varphi^M[a'_i]_i \), and so \( \varphi^M \) is well-defined. As each \( \varphi_i \) is continuous, it follows that \( \varphi^M \) is induced by a continuous map \( \prod_n M_n \to \prod_n M'_n \), and thus is continuous.

Now for \( \alpha = [a_i], \beta = [b_i] \in \mathcal{U}(\mathcal{M}) \) and \( r \in R \), as \( \varphi_i \) is an \( R \)-algebra homomorphism for \( \mathfrak{F} \)-many \( i \), we get

\[
\begin{align*}
\varphi^M(\alpha + \beta) &= \varphi^M[a_i + b_i]_i = [\varphi_i(a_i + b_i)]_i = [\varphi_i(a_i) + \varphi_i(b_i)]_i = \varphi^M(\alpha) + \varphi^M(\beta) \\
\varphi^M(\alpha \cdot \beta) &= \varphi^M[a_i \cdot b_i]_i = [\varphi_i(a_i \cdot b_i)]_i = [\varphi_i(a_i) \cdot \varphi_i(b_i)]_i = \varphi^M(\alpha) \cdot \varphi^M(\beta) \\
\varphi^M(ra) &= \varphi^M[r a_i]_i = [\varphi_i(r a_i)]_i = [r \varphi_i(a_i)]_i = r \varphi^M(\alpha) \\
\varphi^M(1) &= \varphi^M[1]_i = [\varphi_i(1)]_i = [1]_i = 1,
\end{align*}
\]

so indeed \( \varphi^M \) is an \( R \)-algebra homomorphism.

If \( \varphi_i = \psi_i \) for \( \mathfrak{F} \)-many \( i \), then by definition we have \( \varphi^M[a_i]_i = [\varphi_i(a_i)]_i = [\psi_i(a_i)]_i = \psi^M[a_i]_i \), and if \( \varphi_i = \id_{M_i} \) for \( \mathfrak{F} \)-many \( i \), then \( \varphi^M[a_i]_i = [\varphi_i(a_i)]_i = [a_i]_i \). So (1) holds.

For (2), simply note that for \( \mathfrak{F} \)-many \( i \), \( \varphi_i \) and \( \psi_i \) simultaneously exist, and so

\[
(\psi^M \circ \varphi^M)[a_i]_i = \psi^M(\varphi^M[a_i]_i) = \psi^M[\varphi_i(a_i)]_i = [\psi_i(\varphi_i(a_i))]_i = (\psi \circ \varphi)^M[a_i]_i.
\]
In general, $\mathcal{U}(\mathcal{M})$ can be a quite complicated object. However in our setup, the $M_n$’s will always be taken to be finite, of uniformly bounded cardinalities. In that case, we have the following:

**Proposition 1.3.** If $M \in \mathcal{C}$ has finite cardinality, the diagonal map $\Delta : M \rightarrow M^\mathcal{U}$ is an isomorphism.

Now assume that $\mathcal{C}$ is the category of abelian groups or rings, or that the ring $R$ is topologically finitely generated (in particular, if it is finite). If $\mathcal{M} = \{M_n\}_{n \geq 1}$ where each $M_n \in \mathcal{C}$ is a finite set, and the cardinalities $\#M_n$ are bounded, then $\mathcal{U}(\mathcal{M})$ is also finite and we have $\mathcal{U}(\mathcal{M}) \cong M_i$ in $\mathcal{C}$ for $\mathfrak{F}$-many $i$.

**Proof.** As $\Delta : M \rightarrow M^\mathcal{U}$ is already an injective $\mathcal{C}$-morphism, it suffices to show that it is surjective. Take any $\alpha = [a_i]_i \in M^\mathcal{U}$. As $M$ is finite, $\bigsqcup_{a \in M} \{i|a_i = a\}$ is a finite partition of $\mathbb{N}$, and so for some $a \in M$, $a_i = a$ for $\mathfrak{F}$-many $i$. But then $\alpha = [a_i]_i = [a]_i = \Delta(a)$, so indeed $\Delta$ is surjective, and hence an isomorphism.

For the second statement, the assumption on $\mathcal{C}$ implies that there are only finitely many isomorphism classes of $\mathcal{C}$-objects of any fixed cardinality $d$. As the $\#M_n$’s are bounded, there are only finitely many distinct cardinalities $\{\#M_n\}_{n \geq 1}$. It thus follows that there are only finitely many isomorphism classes of $\mathcal{C}$-objects in $\mathcal{M}$.

Thus we may pick some $M \in \mathcal{C}$ (which is necessarily finite) for which $M \cong M_i$ for $\mathfrak{F}$-many $i$. Fix isomorphisms $\varphi_i : M \cong M_i$ for $\mathfrak{F}$-many $i$, and define $\mathfrak{F}$-morphisms $\varphi : M \rightarrow \mathcal{M}$ and $\psi : \mathcal{M} \rightarrow M$ by $\varphi = \{\varphi_i\}$ and $\psi = \{\varphi_i^{-1}\}$. It follows from Proposition 1.2 that $\psi^\mathcal{U} = (\varphi^\mathcal{U})^{-1}$ and so $\varphi^\mathcal{U} : M^\mathcal{U} = \mathcal{U}(M) \rightarrow \mathcal{U}(\mathcal{M})$ is an isomorphism.

Combining this with the first claim, we indeed get $\mathcal{U}(\mathcal{M}) \cong M^\mathcal{U} \cong M \cong M_i$ for $\mathfrak{F}$-many $i$. □

In the case when $\mathcal{C}$ is taken to be the category of $R$-modules (or $R$-algebras), the construction of $\mathcal{U}(\mathcal{M})$ can be reformulated as a localization of modules, and is thus quite well behaved. We finish this section by discussing this situation.

For the reminder of this section, $R$ will denote a finite local ring with maximal ideal $m_R$ and residue field $\mathbb{F} = R/m_R$.

We will let $\mathcal{R} := \prod_{n \geq 1} R$, treated as an $R$-algebra via the diagonal embedding $r \mapsto (r, r, \ldots)$. Proposition 1.1 implies that the natural map $\pi : \mathcal{R} \rightarrow \mathcal{R}^\mathcal{U} = R$ is a surjective ring homomorphism.

Also for any $I \subseteq \mathbb{N}$, we will let $\mathcal{R}_I := \prod_{i \in I} R$, viewed as a quotient of $\mathcal{R}$ via the map $\pi_I : (r_n)_{n \geq 1} \mapsto (r_i)_{i \in I}$. Note that $\pi : \mathcal{R} \rightarrow R$ factors through $\pi_I$ for each $I \in \mathfrak{F}$.

The key observation is that $\pi$ may be viewed as a localization map:

**Proposition 1.4.** View $R$ as a $\mathcal{R}$-algebra via the map $\pi : \mathcal{R} \rightarrow R$. There is a unique prime ideal $\mathfrak{F}_R \in \text{Spec} \mathcal{R}$ for which the $\mathcal{R}$-algebra localization map $R \rightarrow R_{\mathfrak{F}_R}$ is an isomorphism. For this $\mathfrak{F}_R$...
we have:

- The map \( \pi_3 : \mathcal{R}_3 \to R \) is an isomorphism.
- For all \( I \in \mathfrak{F} \) the map \( \pi_{I,3} : \mathcal{R}_3 \to \mathcal{R}_{I,3} \), induced by \( \pi_I : \mathcal{R} \to \mathcal{R}_I \) is an isomorphism.

We will call \( \mathfrak{F}_3 \) the prime (of \( \mathcal{R} \)) associated to \( \mathfrak{F} \).

Finally, if \( \psi : R \to R' \) is a surjection of local rings, inducing the surjection \( \Psi : \mathcal{R} \to \mathcal{R}' := \prod_{n \geq 1} R' \), and \( \mathfrak{F}_3' \in \text{Spec} \mathcal{R}' \) is the prime associated to \( \mathfrak{F} \), then \( \mathfrak{F}_3 = \Psi^{-1}(\mathfrak{F}_3') \).

**Proof.** Assume that there is some \( \mathfrak{F}_3 \in \text{Spec} \mathcal{R} \) which makes \( R \to \mathcal{R}_3 \) into an isomorphism. Clearly we must have \( \ker(\pi : \mathcal{R} \to R) \subseteq \mathfrak{F}_3 \), or we would have \( \mathcal{R}_3 = 0 \). Thus \( \mathfrak{F}_3 = \pi^{-1}(P) \) for some \( P \in \text{Spec} \mathcal{R} \) and \( R_P \cong \mathcal{R}_3 \). But now as \( R \) is a local ring, \( R \to R_P \) is an isomorphism if and only if \( P = m_U \mathcal{R} \). Thus the unique prime \( \mathfrak{F}_3 \) satisfying the condition is \( \mathfrak{F}_3 = \pi^{-1}(m_U \mathcal{R}) \).

We now show that the map \( \pi_3 : \mathcal{R}_3 \to R \) is an isomorphism. As localization is exact, it is surjective.

Take any \( \frac{r}{s} \in \ker(\pi_3) \) where \( r = (r_1, r_2, \ldots) \in \mathcal{R} \). Then \( r \in \ker(\pi) \) so that \( [r_i]_i = 0 \) in \( R \), and hence \( r_i = 0 \) for \( \mathfrak{F} \)-many \( i \). Define \( e = (e_1, e_2, \ldots) \in \mathcal{R} \) by \( e_i = 1 \) if \( r_i = 0 \) and \( e_i = 0 \) if \( r_i \neq 0 \), and note that \( er = 0 \). But by definition \( e_i = 1 \) for \( \mathfrak{F} \)-many \( i \), and so \( \pi(e) = 1 \not\in m_u \mathcal{R} \). Hence \( e \not\in \mathfrak{F}_3 \), and so \( \frac{e}{1} \) is a unit in \( \mathcal{R}_3 \). As \( \frac{e}{1} \frac{r}{s} = 0 \), this implies that \( \frac{r}{s} = 0 \). Therefore \( \ker(\pi_3) = 0 \) and so indeed, \( \pi_3 \) is an isomorphism.

Now for any \( I \in \mathfrak{F} \), \( \pi : \mathcal{R} \to R \) is a composition of surjections \( \pi_I : \mathcal{R} \to \mathcal{R}_I \) and \( \mathcal{R}_I \to R \), and so \( \pi_3 : \mathcal{R}_3 \to R \) is a composition of the surjections \( \pi_{I,3} : \mathcal{R}_3 \to \mathcal{R}_{I,3} \) and \( \mathcal{R}_{I,3} \to R \). So as \( \pi_3 \) is an isomorphism, the latter two maps are isomorphisms as well.

The last statement follows from the commutative diagram

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\pi} & R \\
\downarrow{\Psi} & & \downarrow{\Psi^U} \\
\mathcal{R}' & \xrightarrow{\pi'} & R'
\end{array}
\]

From now on we will always use \( \mathfrak{F}_3 \) to denote the prime of \( \mathcal{R} \) associated to \( \mathfrak{F} \), or just \( \mathfrak{F} \) if \( R \) is clear from context.

We will now investigate ultraproducts of \( R \)-modules (and \( R \)-algebras). Let \( \mathcal{M} = \{M_n\}_{n \geq 1} \) be any sequence of \( R \)-modules, and write \( \mathcal{M} = \prod_{n \geq 1} M_n \) with its natural \( \mathcal{R} \)-module structure. We claim
that the natural surjection \( \pi^\# : \mathcal{M} \to \mathcal{U}(\mathcal{M}) \) is an \( \mathcal{R} \)-module homomorphism, where the \( \mathcal{R} \)-action on \( \mathcal{U}(\mathcal{M}) \) is given by \( \pi : \mathcal{R} \to \mathcal{R} \).

Indeed for any \( r = (r_1, r_2, \ldots) \in \mathcal{R} \) and \( m = (m_1, m_2, \ldots) \in \mathcal{M} \) we have \( r_i = \pi(r) \) for \( \mathfrak{F} \)-many \( i \), and so

\[
\pi^\# (rm) = [r_im_i]_i = [\pi(r)m_i]_i = \pi(r)[m_i]_i = \pi(r)\pi^\# (m).
\]

If additionally the \( M_n \)'s are \( A \)-algebras, then \( \mathcal{U}(\mathcal{M}) \) is an \( \mathcal{R} \)-algebra, and the above morphism is of \( \mathcal{R} \)-algebras.

Proposition 1.4 now allows us to re-interpret \( \pi^\# \) as a localization map of \( \mathcal{R} \)-modules:

**Proposition 1.5.** Let \( \mathcal{M} = \{M_n\}_{n \geq 1} \) be a collection of \( R \)-modules and let \( \mathcal{M} \) and \( \pi^\# : \mathcal{M} \to \mathcal{U}(\mathcal{M}) \) be as above. We have the following:

1. The map \( \pi_3^\# : \mathcal{M}_3 \to \mathcal{U}(\mathcal{M})_3 = \mathcal{U}(\mathcal{M}) \) is an isomorphism of \( \mathcal{R}_3 \) = \( \mathcal{R} \)-modules. If each \( M_n \) is an \( \mathcal{R} \)-algebra then \( \pi_3^\# \) is an isomorphism of \( \mathcal{R} \)-algebras.

2. If \( \varphi = \{\varphi_i\}_{i \in I} : \mathcal{M} \to \mathcal{M}' \) (for \( I \in \mathfrak{F} \)) is a \( \mathfrak{F} \)-morphism of sequences of \( R \)-modules, then the map \( \varphi^\#: \mathcal{U}(\mathcal{M}) \to \mathcal{U}(\mathcal{M}') \) from Proposition 1.1 is the localization of the map

\[
\Phi_I := \prod_{i \in I} \varphi_i : \prod_{i \in I} M_i \to \prod_{i \in I} M_i'
\]

at \( \mathfrak{F} \).

3. The functor \( \mathcal{M} \to \mathcal{U}(\mathcal{M}) \) (from the category of sequences of \( R \)-modules, to the category of \( R \)-modules) is exact.

**Proof.** As localization is exact, \( \pi_3^\# \) is surjective. Now arguing as in Proposition 1.4, if \( m \in \ker(\pi_3^\#) \) where \( m = (m_1, m_2, \ldots) \in \mathcal{M}_3 \) then \( [m_i]_i = 0 \) in \( \mathcal{U}(\mathcal{M})_3 \) and hence \( m_i = 0 \) for all \( i \in I \) for some \( I \in \mathfrak{F} \). But then \( m \in \ker(\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_I) \) and so \( m \in \ker(\mathcal{M}_3 \to \mathcal{M}_3 \otimes_{\mathcal{R}} \mathcal{R}_I, 3 = \mathcal{M}_3) = 0 \). So indeed, \( \ker(\pi_3^\#) = 0 \), and so \( \pi_3^\# \) is an isomorphism of \( R \)-modules. If each \( M_n \) is an \( R \)-algebra then \( \pi_3^\# \) is also a homomorphism of \( R \)-algebras, and thus is an isomorphism of \( R \)-algebras. This proves (1).

For (2), note that \( \mathcal{M}_I := \prod_{i \in I} \varphi_i : \prod_{i \in I} M_i = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_I \), and so \( \mathcal{M}_{I,3} = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_{I,3} = \mathcal{M}_3 \), and similarly for \( \mathcal{M}'_I := \prod_{i \in I} M_i' \). (2) then follows from localizing the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}_I & \xrightarrow{\pi^\#} & \mathcal{U}(\mathcal{M}) \\
\Phi_I \downarrow & & \downarrow \varphi^\# \\
\mathcal{M}'_I & \xrightarrow{\pi^\#} & \mathcal{U}(\mathcal{M}')
\end{array}
\]
Finally, (3) follows by noting that the functors \( \{M_n\}_{n \geq 1} \mapsto \prod_{n \geq 1} M_n \) and \( \mathcal{M} \mapsto M_3 \) are both exact. \(\square\)

# 2 The patching construction

We are now ready to give the main patching construction. Fix a complete DVR \(\mathcal{O}\) with uniformizer \(\lambda\) and finite residue field \(\mathbb{F} = \mathcal{O}/\lambda\) of characteristic \(\ell\). Also fix some \(d \geq 1\), and consider the ring:

\[ S_\infty := \mathcal{O}[[t_1, \ldots, t_d]]. \]

And let \(n = (t_1, \ldots, t_d) \subseteq S_\infty\). Note that \(S_\infty\) is a compact topological ring, so that \(S_\infty/\mathfrak{a}\) is finite for all open ideals \(\mathfrak{a} \subseteq S_\infty\).

The patching construction will take a sequence \(\mathcal{M} = \{M_n\}_{n \geq 1}\) of finite type \(S_\infty\)-modules satisfying certain properties, and produce a reasonably well behaved module \(\mathcal{P}(\mathcal{M})\), which can be roughly thought of as a “limit” of the \(M_n\)’s.

We first make a precise definition of the sequences of \(S_\infty\)-modules we will consider:

**Definition 2.1.** Let \(\mathcal{M} = \{M_n\}_{n \geq 1}\) be a sequence of finite type \(S_\infty\)-modules.

- We say that \(\mathcal{M}\) is a weak patching system if the \(S_\infty\)-ranks of the \(M_n\)’s are uniformly bounded.
- We say that \(\mathcal{M}\) is a patching system if it is a weak patching system, and for any open ideal \(\mathfrak{a} \subseteq S_\infty\), we have \(\text{Ann}_{S_\infty}(M_\mathfrak{a}) \subseteq \mathfrak{a}\) for all but finitely many \(n\).
- We say that \(\mathcal{M}\) is free if \(M_n\) is free over \(S_\infty/\text{Ann}_{S_\infty}(M_n)\) for all but finitely many \(n\).
- If \(M_0\) is a finite type \(\mathcal{O}\)-module, we say that \(\mathcal{M}\) is a patching system over \(M_0\) if \(\mathcal{M}\) is a patching system and we have \(M_n/\mathfrak{a} \cong M_0\) for all but finitely many \(n\).

Furthermore, assume that \(\mathcal{R} = \{R_n\}_{n \geq 1}\) is a sequence of finite type \(S_\infty\)-algebras.

- We say that \(\mathcal{R} = \{R_n\}_{n \geq 1}\) is a (weak) patching algebra, if it is a (weak) patching system.
- We say that \(\mathcal{R}\) is a patching algebra over \(R_0\) if \(R_n/\mathfrak{a} \cong R_0\) as \(\mathcal{O}\)-algebras for all but finitely many \(n\).
- If \(M_n\) is an \(R_n\)-module (viewed as an \(S_\infty\)-module via the \(S_\infty\)-algebra structure on \(R_n\)) for all \(n\) we say that \(\mathcal{M} = \{M_n\}_{n \geq 1}\) is a (weak) patching \(\mathcal{R}\)-module if it is a (weak) patching system.
- If \(\mathcal{R}\) is a patching algebra over \(R_0\) and \(M_0\) is an \(R_0\)-module, we say that \(\mathcal{M}\) is a patching \(\mathcal{R}\)-module over \(M_0\) if \(M_n/\mathfrak{a} \cong M_0\) as \(R_n/\mathfrak{a} \cong R_0\)-modules for all but finitely many \(n\).

Note that the collection of weak patching systems is an abelian category, with the obvious notion of morphism, as is the category of weak patching \(\mathcal{R}\)-modules for any weak patching algebra \(\mathcal{R}\).

For any weak patching system \(\mathcal{M}\) and any ideal \(J \subseteq S_\infty\), we will write \(\mathcal{M}/J := \{M_n/J\}_{n \geq 1}\).

If \(\mathfrak{a} \subseteq S_\infty\) is open, note that each \(M_n/\mathfrak{a}\) is a finite type \(S_\infty/\mathfrak{a}\)-module and the ranks of the \(M_n/\mathfrak{a}\)’s are bounded. As \(S_\infty/\mathfrak{a}\) is finite, it follows that each \(M_n/\mathfrak{a}\) is finite, and the cardinalities of the
$M_n/a$’s are bounded. Proposition 1.3 then implies that $U(\mathcal{M}/a) \cong M_i/a$ as $S_\infty/a$-modules (and hence as $S_\infty$-modules) for $\exists$-many $i$.

Now for any $a' \subseteq a$, the surjections $M_n/a' \twoheadrightarrow M_n/a$ induce a surjection $U(\mathcal{M}/a') \twoheadrightarrow U(\mathcal{M}/a)$. In fact, by the exactness of $U(-)$, this surjection induces an isomorphism $U(\mathcal{M}/a')/a \cong U(\mathcal{M}/a)$ of $S_\infty$-modules (or $S_\infty$-algebras if $\mathcal{M}$ is a weak patching algebra).

Thus the $U(\mathcal{M}/a)$’s form an inverse system, and so we may make the following definition:

**Definition 2.2.** For any weak patching system $\mathcal{M}$ define:

$$\mathcal{P}(\mathcal{M}) := \lim_{\leftarrow a} U(\mathcal{M}/a).$$

As $U(-)$ is functorial, $\mathcal{P}(-)$ is functorial as well. For a morphism $f : \mathcal{M} \to \mathcal{N}$ of weak patching systems, let $f^\mathcal{P} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ denote the induced map.

Note that as each $U(\mathcal{M}/a)$ is an $S_\infty$-module, $\mathcal{P}(\mathcal{M})$ is an $S_\infty$-module. Similarly if $\mathcal{R}$ is a weak patching algebra then $\mathcal{P}(\mathcal{R})$ is an $S_\infty$-algebra, and if $\mathcal{M}$ is a weak patching $\mathcal{R}$-module then $\mathcal{P}(\mathcal{M})$ is a $\mathcal{P}(\mathcal{R})$-module (with its $S_\infty$-module structure induced from the $S_\infty$-algebra structure on $\mathcal{P}(\mathcal{R})$).

In turns out that $\mathcal{P}$ is a reasonably well behaved functor. In fact:

**Proposition 2.3.** $\mathcal{P}$ is a right-exact functor.

**Proof.** Let $\textbf{Ab}$ be the category of abelian groups. For any index set $I$, let $\textbf{finAb}^I$ be the category of inverse systems of finite abelian groups indexed by $I$. We claim the the functor $\lim_{\leftarrow a} : \textbf{finAb}^I \to \textbf{Ab}$ is exact.

By [Sta18, Tag 0598], it suffices to show that any $(A_i, f_{ji} : A_j \to A_i) \in \textbf{finAb}^I$ satisfies the Mittag-Leffler condition: For any $i \in I$ there is a $j \geq i$ for which $\text{im}(f_{ki}) = \text{im}(f_{ji})$ for all $k \geq j$.

But as $A_i$ is finite, it has only finitely many subgroups and so the collection $\{\text{im}(f_{ji})\}_{j \geq i}$ of subgroups of $A_i$ must have some minimal member, $\text{im}(f_{ji})$, under inclusion. Then for any $k \geq j$, $\text{im}(f_{ki}) = \text{im}(f_{ji} \circ f_{kj}) \subseteq \text{im}(f_{ji})$ and hence $\text{im}(f_{ki}) = \text{im}(f_{ji})$. So indeed every object of $\textbf{finAb}^I$ satisfies the Mittag-Leffler condition, and so $\lim_{\leftarrow a}$ is exact.

Now assume $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are weak patching systems, and we have an exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

Then for any $a \subseteq S_\infty$, $\mathcal{A}/a \to \mathcal{B}/a \to \mathcal{C}/a \to 0$ is right-exact, so by the exactness of $U(-)$ we get the right exact sequence

$$U(\mathcal{A}/a) \to U(\mathcal{B}/a) \to U(\mathcal{C}/a) \to 0.$$

Thus we have a right-exact sequence of complexes

$$\left(U(\mathcal{A}/a)\right)_a \to \left(U(\mathcal{B}/a)\right)_a \to \left(U(\mathcal{C}/a)\right)_a \to 0.$$
But now as \( U(\mathcal{A}/a), U(\mathcal{B}/a) \) and \( U(\mathcal{C}/a) \) are all finite, the above argument shows that taking inverse limits preserves exactness, and so indeed

\[
\mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{C}) \to 0
\]

is right-exact. \( \square \)

Note that in general, \( \mathcal{P} \) is not left-exact. Indeed, letting \( \mathcal{M} = \{S_n\}_{n \geq 1} \) and defining \( \varphi : \mathcal{M} \to \mathcal{M} \) via \( \varphi_n(s) = \lambda^n s \) we see that \( \mathcal{P}(\mathcal{M}) = S_\infty \) and \( \varphi^\mathcal{P} : S_\infty \to S_\infty \) is the zero map (since for any \( a \), \( \varphi_{n,a} : S_\infty/a \to S_\infty/a \) is the zero map for all but finitely many \( n \)).

Now Proposition 2.3, and Definition 2.1 easily imply the following basic properties:

**Proposition 2.4.** If \( \mathcal{M} = \{M_n\}_{n \geq 1} \) is a patching system then:

1. \( \mathcal{P}(\mathcal{M}) \) is a finite type, faithful \( S_\infty \)-module.
2. For any open ideal \( a \subseteq S_\infty \), \( \mathcal{P}(\mathcal{M})/a \cong U(\mathcal{M}/a) \).
3. If \( \mathcal{M} \) is a patching system over \( M_0 \), then \( \mathcal{P}(\mathcal{M})/n \cong M_0 \).
4. If \( \mathcal{M} \) is free, then \( \mathcal{P}(\mathcal{M}) \) is a finite type free \( S_\infty \)-module.
5. If \( M \) is a finite type \( S_\infty \)-module and \( \mathcal{M} = \{M_n\}_{n \geq 1} \) is the constant weak patching system, then there is a natural isomorphism \( \mathcal{P}(\mathcal{M}) \cong M \).

**Proof.** First, for (5), \( M/a \) is finite for all open \( a \subseteq S_\infty \) and so Proposition 1.3 gives a natural isomorphism \( (M/a)^\mathcal{M} \cong M/a \). Thus

\[
\mathcal{P}(\mathcal{M}) = \lim_{\mathcal{M}} U(M/a) = \lim_{\mathcal{M}} U(M/a)^\mathcal{M} = \lim_{\mathcal{M}} M/a \cong M
\]

as any finite type \( S_\infty \)-module is complete.

Now if \( J \subseteq S_\infty \) is any ideal, then Proposition 2.3 implies that \( \mathcal{P}(\mathcal{M})/J \cong \mathcal{P}(\mathcal{M}/J) \).

Letting \( J = a \subseteq S_\infty \) open, we have \( (\mathcal{M}/a)/a' \cong \mathcal{M}/a \) for all \( a' \subseteq a \) and so \( U((\mathcal{M}/a)/a') \cong U(\mathcal{M}/a) \).

Thus we have

\[
\mathcal{P}(\mathcal{M})/a \cong \mathcal{P}(\mathcal{M}/a) = \lim_{a' \subseteq a} U((\mathcal{M}/a)/a') = \lim_{a' \subseteq a} U((\mathcal{M}/a)/a') \cong \lim_{a' \subseteq a} U(\mathcal{M}/a) \cong U(\mathcal{M}/a),
\]

proving (2).

Assuming that \( \mathcal{M} \) is a patching system over \( M_0 \) and letting \( J = n \), we get

\[
\mathcal{P}(\mathcal{M})/n \cong \mathcal{P}(\mathcal{M}/n) \cong \mathcal{P}(M_0) \cong M_0,
\]

proving (3).

Now as the \( S_\infty \)-ranks of of the \( M_n \)'s are bounded, say by some \( N \geq 1 \), there is a surjection of patching systems \( S_\infty^N \to \mathcal{M} \) and hence a surjection \( S_\infty^N = \mathcal{P}(S_\infty^N) \to \mathcal{P}(\mathcal{M}) \). Thus \( \mathcal{P}(\mathcal{M}) \) is a finite type \( S_\infty \)-module.
Now for any open \( a \subseteq S_\infty \), we have \( \text{Ann}_{S_\infty}(M_n) \subseteq a \) for all but finitely many \( n \), by assumption, and hence \( \text{Ann}_{S_\infty}(M_n/a) = a \) for all such \( n \). Thus \( \text{Ann}_{S_\infty}(U(\mathcal{M}/a)) = a \). But now we have

\[
\text{Ann}_{S_\infty}(\mathcal{P}(\mathcal{M})) \subseteq \text{Ann}_{S_\infty}(\mathcal{P}(\mathcal{M})/a) = \text{Ann}_{S_\infty}(U(\mathcal{M}/a)) = a
\]

for all \( a \), and so \( \text{Ann}_{S_\infty}(\mathcal{P}(\mathcal{M})) = 0 \), proving (1).

Lastly, assume that \( \mathcal{M} \) is free. Then for all but finitely many \( n \), \( M_n \cong (S_\infty/\text{Ann}_{S_\infty}(M_n))^{r_n} \) for some \( r_n \). As there \( r_n \)'s are bounded, there is some \( r \) such that \( r_i = r \), and hence \( M_i \cong (S_\infty/\text{Ann}_{S_\infty}(M_i))^{r_i} \), for \( \mathcal{J} \)-many \( i \).

Define an \( \mathcal{J} \)-morphism \( \varphi : S_\infty^{r} \to \mathcal{M} \) by letting \( \varphi_i : S_\infty^{r} \to (S_\infty/\text{Ann}_{S_\infty}(M_i))^{r_i} \cong M_i \) for \( \mathcal{J} \)-many \( i \). Then for any open \( a \subseteq S_\infty \), \( \varphi_i : S_\infty^{r}/a \to M_i/a \) is an isomorphism for \( \mathcal{J} \)-many \( i \), and so \( \varphi \) induces an isomorphism \( U((S_\infty/a)^{r}) \cong U(\mathcal{M}/a) \) for all \( a \), and thus an isomorphism \( \varphi^\mathcal{P} : S_\infty^{r} = \mathcal{P}(S_\infty^{r}) \to \mathcal{P}(\mathcal{M}) \) is an isomorphism. So indeed, \( \mathcal{P}(\mathcal{M}) \) is a finite type, free \( S_\infty \)-module, proving (4).

The following simple consequence of Proposition 2.4 is central to most applications of this theory:

**Corollary 2.5.** If \( \mathcal{R} \) is a weak patching algebra and \( \mathcal{M} \) is a patching \( \mathcal{R} \)-module, then the homomorphism \( S_\infty \to \mathcal{P}(\mathcal{R}) \) inducing the \( S_\infty \)-algebra structure on \( \mathcal{P}(\mathcal{R}) \) is injective and the Krull dimension of \( \mathcal{P}(\mathcal{R}) \) is \( d + 1 = (\dim S_\infty) \).

If, furthermore, \( \mathcal{M} \) is a free patching \( \mathcal{R} \)-module then \( \mathcal{P}(\mathcal{M}) \) is a maximal Cohen–Macaulay module over \( \mathcal{P}(\mathcal{R}) \).

Finally if \( \mathcal{P}(\mathcal{R}) \) is a Cohen–Macaulay ring, then \( (t_1, \ldots, t_d, \lambda) \subseteq S_\infty \subseteq \mathcal{P}(\mathcal{R}) \) is a regular sequence for \( \mathcal{P}(\mathcal{R}) \).

**Proof.** For the first statement, the map \( \iota : S_\infty \to \mathcal{P}(\mathcal{R}) \) induces the \( S_\infty \)-module structure on \( \mathcal{P}(\mathcal{M}) \), which is *faithful* by Proposition 2.4(1), and so \( \iota \) must be injective.

As \( \mathcal{P}(\mathcal{R}) \) is of finite type over \( S_\infty \), it follows that the Krull dimension of \( \mathcal{P}(\mathcal{R}) \) is just \( \dim \mathcal{P}(\mathcal{R}) = \dim \iota(S_\infty) = \dim S_\infty = d + 1 \).

Now assume that \( \mathcal{M} \) is free. By Proposition 2.4(4), \( \mathcal{P}(\mathcal{M}) \) is finite free over \( \iota(S_\infty) \cong S_\infty \) and so \( \mathcal{P}(\mathcal{M}) \) is indeed Cohen–Macaulay of dimension \( d + 1 = \dim S_\infty \). In particular \( (t_1, \ldots, t_d, \lambda) \subseteq S_\infty \subseteq \mathcal{P}(\mathcal{R}) \) is a regular sequence.

Now if \( \mathcal{P}(\mathcal{R}) \) is also Cohen–Macaulay, then a sequence is regular for \( \mathcal{P}(\mathcal{M}) \) if and only if it is regular for \( \mathcal{P}(\mathcal{R}) \), so the last statement follows.

### 3 Covers of Patching Algebras

In the classical setup of Taylor–Wiles–Kisin patching, one considers a patching algebra \( \mathcal{R} = \{ R_n \}_{n \geq 1} \), where the \( R_n \)'s are all taken to be quotients of a fixed ring \( R_\infty \). We thus make a
the following definition:

**Definition 3.1.** If $\mathcal{R} = \{R_n\}_{n \geq 1}$ is a weak patching algebra we say that a cover $(R_\infty, \{\varphi_n\})$ of $\mathcal{R}$ is:

- A complete, Cohen–Macaulay ring $R_\infty$, which is topologically finitely generated as an $O$-algebra, of Krull dimension $d + 1 (= \dim S_\infty)$ together with:
- For each $n$, a continuous, surjective $O$-algebra homomorphism $\varphi_n : R_\infty \to R_n$.

We will often use $R_\infty$ to denote the cover $(R_\infty, \{\varphi_n\})$.

Note that:

**Lemma 3.2.** If $(R_\infty, \{\varphi_n\})$ is a cover of a weak patching algebra $\mathcal{R}$, then the $\varphi_n$’s induce a natural continuous surjection $\varphi_\infty : R_\infty \to P(\mathcal{R})$.

**Proof.** The $\varphi_n$’s induce a continuous map $\Phi = \prod_{n \geq 1} \varphi_n : R_\infty \to \prod_{n \geq 1} R_n$, and thus induce continuous maps

$$
\Phi_a : R_\infty \xrightarrow{\Phi} \prod_{n \geq 1} R_n \to \prod_{n \geq 1} (R_n/a) \to U(\mathcal{R}/a)
$$

for all open $a \subseteq S_\infty$, and thus they indeed induce a continuous map

$$
\varphi_\infty = (\Phi_a)_a : R_\infty \to \lim_{\rightarrow a} U(\mathcal{R}/a) = P(\mathcal{R}).
$$

We now claim that each $\Phi_a$ is surjective. As each map $R_\infty \xrightarrow{\varphi_n} R_n \to R_n/a$ is continuous, we may give each $R_n/a$ the structure of a continuous $R_\infty$-algebra. Then the map $\Phi_a : R_\infty \to U(\mathcal{R}/a)$ defines the continuous $R_\infty$-algebra structure on $U(\mathcal{R}/a)$ from Proposition 1.1. By Proposition 1.3, $U(\mathcal{R}/a) \cong R_i/a$ as $R$-algebras for $\mathfrak{f}$-many $i$. But for any such $i$, the map $R_\infty \xrightarrow{\varphi_i} R_i \to R_i/a$ defining the $R_\infty$-algebra structure is surjective, and so $\Phi_a : R_\infty \to U(\mathcal{R}/a)$ must indeed be surjective.

It follows that $\varphi_\infty(R_\infty) \subseteq P(\mathcal{R})$ is dense. But now as $R_\infty$ is topologically finitely generated over $O$, it is compact, and so $\varphi_\infty(R_\infty)$ is also closed in $P(\mathcal{R})$. Therefore $\varphi_\infty$ is indeed surjective.

We will say that the cover $R_\infty$ is **minimal** if $\varphi_\infty$ is an isomorphism.

Lemma 3.2 and Corollary 2.5 give the following useful result:

**Corollary 3.3.** If $\mathcal{R}$ is a weak patching algebra with a cover $R_\infty$, and $\mathcal{M}$ is any free patching $\mathcal{R}$-module, then $P(\mathcal{M})$ is maximal Cohen–Macaulay over $R_\infty$.

**Proof.** By Lemma 3.2, $P(\mathcal{R})$ may be thought of as a quotient of $R_\infty$. From the definition of Cohen–Macaulay modules, if $f : A \to B$ is any surjective map of rings, and $M$ is a $B$-module, then $M$ is Cohen–Macaulay over $B$ if and only if it is Cohen–Macaulay over $A$. Thus by Corollary 2.5, $P(\mathcal{M})$ is Cohen–Macaulay over $R_\infty$.
We can now prove the main result of this appendix:

**Theorem 3.4.** Let $R_0$ be a finite type $\mathcal{O}$-algebra and let $M_0$ be a nonzero $R_0$-module, which is finite type and free over $\mathcal{O}$. Assume that we are given:

- A weak patching algebra $\mathcal{R} = \{R_n\}_{n \geq 1}$ over $R_0$;
- A free patching $\mathcal{R}$-module $\mathcal{M} = \{M_n\}_{n \geq 1}$ over $M_0$;
- A cover $R_\infty$ of $\mathcal{R}$, where $R_\infty$ is a domain.

Then we have the following:

1. $R_\infty$ is a minimal cover.
2. $R_\infty \cong \mathcal{P}(\mathcal{R})$ is Cohen–Macaulay, $(t_1, \ldots, t_d, \lambda) \subseteq R_\infty$ is a regular sequence for $R_\infty$, $R_\infty/n \cong R_0$ and $R_0$ is Cohen–Macaulay and $\lambda$-torsion free.
3. If $\eta$ is any generic point of $\text{Spec } R_0$, with function field $\kappa(\eta)$ (i.e. $\kappa(\eta)$ is the field of fractions of $R_0/\eta$), then

   \[
   \dim_{\kappa(\eta)} M_0 \otimes_{R_0} \kappa(\eta) \geq \dim_{K(R_\infty)} \mathcal{P}(\mathcal{M}) \otimes_{R_\infty} K(R_\infty) \geq 1,
   \]

   where $K(R_\infty)$ is the fraction field of $R_\infty$. In particular, $\dim_{K(\eta)} M_0 \otimes_{R_0} \kappa(\eta)$ is always nonzero.

**Proof.** Let $\varphi_\infty : R_\infty \rightarrow \mathcal{P}(\mathcal{R})$ be the map from Lemma 3.2. If $\varphi_\infty$ is not injective, then $\dim \mathcal{P}(\mathcal{R}) = \dim R_\infty/\ker \varphi_\infty < \dim R_\infty = d + 1$, contradicting Corollary 2.5, and so $\varphi_\infty$ is indeed an isomorphism, proving (1).

Now $\mathcal{P}(\mathcal{R}) \cong R_\infty$ is Cohen–Macaulay by our assumptions on $R_\infty$, so Corollary 2.5 implies that $(t_1, \ldots, t_d, \lambda) \subseteq R_\infty$ is an $R_\infty$-regular sequence. Also Proposition 2.4 implies that $R_\infty/(t_1, \ldots, t_d) = R_\infty/n \cong R_0$. Now by the definition of regular sequences, it follows that $R_0$ is Cohen–Macaulay and $(\lambda)$ is an $R_0$-regular sequence, which implies that $R_0$ is $\lambda$-torsion free. This proves (2).

It remains to prove (3). Let $K(S)$ be the fraction field of $S_\infty$. Then $R_\infty \otimes_{S_\infty} K(S_\infty)$ is a finite-dimensional $K(S_\infty)$ algebra which is still a domain, so it follows that $R_\infty \otimes_{S_\infty} K(S_\infty)$ is a field. In particular $R_\infty \otimes_{S_\infty} K(S_\infty) \cong K(R_\infty)$. It follows from this that

\[
\mathcal{P}(\mathcal{M}) \otimes_{R_\infty} K(R_\infty) \cong \mathcal{P}(\mathcal{M}) \otimes_{S_\infty} K(S_\infty).
\]

As $\mathcal{P}(\mathcal{M})$ is free over $S_\infty$ (and nonzero) it follows that $\mathcal{P}(\mathcal{M}) \otimes_{R_\infty} K(R_\infty) \neq 0$ and so

\[
\dim_{K(R_\infty)} \mathcal{P}(\mathcal{M}) \otimes_{R_\infty} K(R_\infty) \geq 1.
\]

For any $P \in \text{Spec } R_\infty$, let $\kappa(P)$ be the residue field of $P$ (that is, the field of fractions of $R_\infty/P$). As $\mathcal{P}(\mathcal{M})$ is a finite type $R_\infty$ algebra, the map $P \mapsto \dim_{\kappa(P)} \mathcal{P}(M) \otimes_{R_\infty} \kappa(P)$ is upper semi-continuous.
on Spec $R_\infty$ and so in particular
\[
\dim_{\kappa(P)} \mathcal{P}(M) \otimes_{R_\infty} \kappa(P) \geq \dim_{K(R_\infty)} \mathcal{P}(\mathcal{M}) \otimes_{R_\infty} K(R_\infty) \geq 1
\]
for any $P$, as Spec $R_\infty$ is irreducible.

Now consider the quotient map $g : R_\infty \to R_\infty/n \cong R_0$ and let $P_\eta = g^{-1}(\eta) \in \text{Spec } R_\infty$. As $n = \ker g$, we get that $n \subseteq P_\eta$. It follows that $R_\infty/P_\eta \cong (R_\infty/n)/(P_\eta/n) \cong R_0/\eta$, so that $\kappa(P_\eta) \cong \kappa(\eta)$ and
\[
\mathcal{P}(\mathcal{M}) \otimes_{R_\infty} \kappa(P_\eta) \cong (\mathcal{P}(\mathcal{M})/n) \otimes_{R_\infty/n} \kappa(\eta) \cong M_0 \otimes_{R_0} \kappa(\eta),
\]
and so the above result gives (3).

\section{Duality}

In order to prove our main result, we will also need to show that patching preserves duality. Over an arbitrary ring, duality can be tricky to define, so we will restrict ourselves to considering patching systems $\mathcal{M} = \{M_n\}_{n \geq 1}$ where the rings $S_\infty/\text{Ann}_{S_\infty}(M_n)$ are well behaved. Specifically we make the following definition:

\textbf{Definition 4.1.} Let $\mathcal{R} = \{R_n\}_{n \geq 1}$ be a weak patching algebra, and let $\mathcal{M} = \{M_n\}_{n \geq 1}$ be a patching $\mathcal{R}$-module. We say that $\mathcal{M}$ is dualizable if it is free and each $S_\infty/\text{Ann}_{S_\infty}(M_n)$ is a local complete intersection\footnote{We could, with a little extra work, weaken this condition to only requiring that $S_\infty/\text{Ann}_{S_\infty}(M_n)$ be Cohen–Macaulay, but as the rings $S_\infty/\text{Ann}_{S_\infty}(M_n)$ are always complete intersections in practice, there is no advantage to doing this.}.

If $\mathcal{M}$ is dualizable, we define its dual to be the patching $\mathcal{R}$-module
\[
\mathcal{M}^* = \{M_n^*\}_{n \geq 1} := \{\text{Hom}_{S_\infty}(M_n, S_\infty/\text{Ann}_{S_\infty}(M_n))\}_{n \geq 1},
\]
where for each $n$, $R_n$ acts on $M_n^* := \text{Hom}_{S_\infty}(M_n, S_\infty/\text{Ann}_{S_\infty}(M_n))$ by $(rf)(x) = f(rx)$.

The following lemma shows that the results of this subsection will be applicable to the specific patching systems we consider in Section ??

\textbf{Lemma 4.2.} Let $d' \leq d$ be an integer and let
\[
S_\infty' := \mathcal{O}[[t_1, \ldots, t_{d'}]] \subseteq \mathcal{O}[[t_1, \ldots, t_d]] = S_\infty.
\]
Assume that for each integer $n \geq 1$ we are given:
\begin{itemize}
  \item A finite type $S_\infty'$-algebra $R_n$, such that the $S_\infty'$-ranks of the $R_n$'s are bounded.
  \item Finite type $R_n$-modules $M_n$ and $N_n$, whose $S_\infty'$-ranks are again bounded.
\end{itemize}
Integers $e(n,1), e(n,2), \ldots, e(n,d') \geq n$ such that $M_n$ and $N_n$ are free over the ring:

$$O[\Delta_n] := S'_\infty / I_n := O[[t_1, \ldots, t_d]] / \frac{(1 + t_1)^{e(n,1)} - 1, (1 + t_2)^{e(n,2)} - 1, \ldots, (1 + t_d)^{e(n,d')} - 1)}$$

(where the $O[\Delta_n]$ action is induced by the $S'_\infty$ actions on $M_n$ and $N_n$).

Define:

$$R_n := R_n[[t_{d'+1}, \ldots, t_d]] = R_n \otimes O[[t_{d'+1}, \ldots, t_d]] = R_n \otimes_{S'_\infty} S_\infty$$

$$M_n := M_n[[t_{d'+1}, \ldots, t_d]] = M_n \otimes O[[t_{d'+1}, \ldots, t_d]] = M_n \otimes_{S'_\infty} S_\infty$$

$$N_n := N_n[[t_{d'+1}, \ldots, t_d]] = N_n \otimes O[[t_{d'+1}, \ldots, t_d]] = N_n \otimes_{S'_\infty} S_\infty$$

and let $\mathcal{R} = \{R_n\}_{n \geq 1}$, $\mathcal{M} = \{M_n\}_{n \geq 1}$ and $\mathcal{N} = \{N_n\}_{n \geq 1}$. Then:

1. $\mathcal{R}$ is a weak patching algebra, and $\mathcal{M}$ and $\mathcal{N}$ are free patching $\mathcal{R}$-algebras.
2. $\mathcal{M}$ and $\mathcal{N}$ are dualizable.

Furthermore, if, for each $n \geq 1$, we are given an $R_n$-equivariant perfect pairing $\langle \ , \rangle : M_n \times N_n \rightarrow O$, then we have $\mathcal{M}^* \cong \mathcal{N}$.

Proof. As the $S'_\infty$-ranks of the $R_n$’s, $M_n$’s and $N_n$’s are bounded, so are the $S_\infty$-ranks of the $R_n$’s, $M_n$’s and $N_n$’s. Thus $\mathcal{R}$ is a weak patching algebra and $\mathcal{M}$ and $\mathcal{N}$ are weak patching $\mathcal{R}$-modules.

The show that $M_n$ and $N_n$ are patching $\mathcal{R}$-modules, we must show that for any open ideal $\mathfrak{a} \subseteq S_\infty$, $\text{Ann}_{S_\infty}(M_n), \text{Ann}_{S_\infty}(N_n) \subseteq \mathfrak{a}$ for all but finitely many $n$. But by assumption, we have $\text{Ann}_{S_\infty}(M_n) = \text{Ann}_{S_\infty}(N_n) = I_n$ for all $n$ (where $I_n$ is now interpreted as an ideal of $S_\infty$). As $S_\infty/\mathfrak{a}$ is finite, and the group $1 + m_{S_\infty}$ is pro-$\ell$, the group $(1 + m_{S_\infty})/\mathfrak{a} := \text{im}(1 + m_{S_\infty} \rightarrow S_\infty \rightarrow S_\infty/\mathfrak{a})$ is a finite $\ell$-group. Since $1 + t_i \in 1 + m_{S_\infty}$ for all $i$, there is an integer $K \geq 0$ such that $(1 + t_i)^{\ell^K} \equiv 1 \pmod{\mathfrak{a}}$ for all $i = 1, \ldots, d'$. Then for any $n \geq K$, $e(n,i) \geq n \geq K$ for all $i$, and so indeed $I_n \subseteq \mathfrak{a}$ by definition.

Also by assumption, $M_n$ and $N_n$ are free over $S'_\infty / I_n = O[\Delta_n]$, and so $M_n$ and $N_n$ are free over $O[\Delta_n] \otimes_{S_\infty} S_\infty = O[\Delta][[t_{d'+1}, \ldots, t_d]] = S_\infty / \text{Ann}_{S_\infty}(M_n) = S_\infty / \text{Ann}_{S_\infty}(N_n)$, so $\mathcal{M}$ and $\mathcal{N}$ are free patching $\mathcal{R}$-modules, proving (1).

For (2), simply note that $O[\Delta_n]$ is a finite free $O$-module, and so $\text{dim } O[\Delta] = 1 = \text{dim } S'_\infty - d'$. As $I_n$ is generated by $d'$ elements, it follows that $O[\Delta_n]$ is local complete intersection. Thus $S_\infty / \text{Ann}_{S_\infty}(M_n) = S_\infty / \text{Ann}_{S_\infty}(N_n) = O[\Delta][[t_{d'+1}, \ldots, t_d]]$ is also a local complete intersection, and so $\mathcal{M}$ and $\mathcal{N}$ are indeed dualizable.

It remains to show that $\mathcal{M}^* \cong \mathcal{N}$, that is that

$$N_n \cong \text{Hom}_{S_\infty}(M_n, S_\infty / I_n) = \text{Hom}_{S_\infty / I_n}(M_n, S_\infty / I_n)$$

as $R_n$-modules for all $n$. As we are given an $R_n$-equivariant perfect pairing $M_n \times N_n \rightarrow O$ we have that $N_n \cong \text{Hom}_O(M_n, O)$ as $R_n$-modules. As $M_n$ is finite free over $O$, it now follows that

$$N_n = N_n \otimes O[[t_{d'+1}, \ldots, t_d]] \cong \text{Hom}_O[[t_{d'+1}, \ldots, t_d]](M_n, O[[t_{d'+1}, \ldots, t_d]])$$
as \( R^\square_n = R_n \otimes_O \mathcal{O}[[t_{d'+1}, \ldots, t_d]] \)-modules.

Now as \( S_\infty/I_n = \mathcal{O}[\Delta_n][[t_{d'+1}, \ldots, t_d]] \) is a local complete intersection of the same dimension as \( \mathcal{O}[[t_{d'+1}, \ldots, t_d]] \), the claim follows from the following commutative algebra lemma.

\[ \square \]

**Lemma 4.3.** If \( A \) is a local Cohen–Macaulay ring and \( B \) is an \( A \)-algebra which is also Cohen–Macaulay with \( \dim A = \dim B \), then for and \( B \)-module \( M \),

\[
\text{Hom}_A(M, \omega_A) \cong \text{Hom}_B(M, \omega_B)
\]

as left \( \text{End}_B(M) \)-modules.

**Proof.** By [Sta18, Tag 08YP] there is an isomorphism

\[
\text{Hom}_A(M, \omega_A) \cong \text{Hom}_B(M, \text{Hom}_A(B, \omega_A))
\]

sending \( \alpha : M \to \omega_A \) to \( \alpha' : m \mapsto (b \mapsto \alpha(bm)) \), which clearly preserves the action of \( \text{End}_B(M) \) (as \( (\alpha \circ \psi)(bm) = \alpha(b\psi(m)) \) for any \( \psi \in \text{End}_B(M) \)). It remains to show that \( \text{Hom}_A(B, \omega_A) \cong \omega_B \), which is just Theorem 21.15 from [Eis95] in the case \( \dim A = \dim B \).

We are now ready to show that patching preserves duality:

**Theorem 4.4.** Let \( \mathcal{R} \) be a weak patching algebra and let \( \mathcal{M} \) be a dualizable patching \( \mathcal{R} \)-module. Then we have \( \mathcal{P}(\mathcal{M}^*) \cong \text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty) \) as \( \mathcal{P}(\mathcal{R}) \)-modules.

Furthermore, if \( R_\infty \) is a cover of \( \mathcal{R} \) (which is assumed to be Cohen–Macaulay by our definition of cover) then \( \mathcal{P}(\mathcal{M}^*) \cong \text{Hom}_{R_\infty}(\mathcal{P}(\mathcal{M}), \omega_{R_\infty}) \) as \( R_\infty \)-modules.

**Proof.** We shall first compute \( \mathcal{U}(\mathcal{M}^*/a) \) for any open ideal \( a \subseteq S_\infty \). For any such \( a \), we have \( \text{Ann}_{S_\infty}(M_n) \subseteq a \) for all but finitely many \( n \), and so \( S_\infty/a \) is a \( S_\infty/\text{Ann}_{S_\infty}(M_n) \) for all such \( n \).

But now for all sufficiently large \( n \), \( M_n \) is finite free over \( S_\infty/\text{Ann}_{S_\infty}(M_n) \) by assumption, and so it is projective. Thus the functor \( \text{Hom}_{S_\infty}(M_n, -) = \text{Hom}_{S_\infty/\text{Ann}_{S_\infty}(M_n)}(M_n, -) \) is exact and so if \( \text{Ann}_{S_\infty}(M_n) \subseteq a \) then

\[
M_n^*/a = \text{Hom}_{S_\infty}(M_n, S_\infty/\text{Ann}_{S_\infty}(M_n))/a \cong \text{Hom}_{S_\infty}(M_n, S_\infty/a) = \text{Hom}_{S_\infty/a}(M_n/a, S_\infty/a)
\]

as \( R_n/a \)-modules.

Now by Proposition 1.3, for \( \mathfrak{F} \)-many \( i \) we have that \( \mathcal{U}(\mathcal{R}/a) \cong R_i/a \) and \( \mathcal{U}(\mathcal{M}/a) \cong M_i/a \) and \( \mathcal{U}(\mathcal{M}^*/a) \cong M_i^*/a \) as \( R_i/a \)-modules. Taking any such \( i \), the above computation gives that

\[
\mathcal{U}(\mathcal{M}^*/a) \cong \text{Hom}_{S_\infty/a}(\mathcal{U}(\mathcal{M}/a), S_\infty/a)
\]

as \( \mathcal{U}(\mathcal{R}/a) \)-modules. Taking inverse limits, it now follows that

\[
\mathcal{P}(\mathcal{M}^*) \cong \lim_{\mathfrak{F}} \text{Hom}_{S_\infty/a}(\mathcal{U}(\mathcal{M}/a), S_\infty/a)
\]

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as $\mathcal{P}(\mathcal{R})$-modules. It remains to show that the right hand side is just $\text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty)$. But using the fact that $\mathcal{P}(\mathcal{M})$, and thus $\text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty)$ is a finite free $S_\infty$-module (and thus is $m_{S_\infty}$-adically complete) we get that

$$\text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty) \cong \lim_{\rightarrow a} \text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty)/a$$

as $\mathcal{P}(\mathcal{R}) = \lim_{\rightarrow a} \mathcal{P}(\mathcal{R})/a$-modules. But now for any $a$, as $\mathcal{P}(\mathcal{M})$ is a finite free, and hence projective, $S_\infty$-module

$$\text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty)/a \cong \text{Hom}_{S_\infty/a}(\mathcal{P}(\mathcal{M})/a, S_\infty/a) \cong \text{Hom}_{S_\infty/a}(U(\mathcal{M}/a), S_\infty/a)$$

as $\mathcal{P}(\mathcal{R})/a = U(\mathcal{R}/a)$-modules. So indeed

$$\text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty) \cong \lim_{\rightarrow a} \text{Hom}_{S_\infty/a}(U(\mathcal{M}/a), S_\infty/a) \cong \mathcal{P}(\mathcal{M}^*)$$

as $\mathcal{P}(\mathcal{R})$-modules.

Now assume that $(R_\infty, \{\varphi_n\}_n)$ is a cover of $\mathcal{R}$ and let $\varphi_\infty : R_\infty \rightarrow \mathcal{P}(\mathcal{R})$ be the map from Lemma 3.2. We first note that we may pick an embedding $i : S_\infty \hookrightarrow R_\infty$, giving $R_\infty$ the structure of an $S_\infty$-algebra, which makes the map $\varphi_\infty$ into an $S_\infty$-algebra homomorphism. Indeed as $\varphi_\infty$ is surjective, for each $i = 1, \ldots, d$ we may simply pick a lift $i(t_i)$ with $\varphi_\infty(i(t_i)) = t_i$ (where we treat $S_\infty$ as a subring of $S_\infty$) so that $\varphi_\infty \circ i = \text{id}$.

It now follows that $\mathcal{P}(\mathcal{M})$ is an $R_\infty$-module, and the $S_\infty$-action on $\mathcal{P}(\mathcal{M})$ is induced by the $S_\infty$-algebra structure on $R_\infty$. But now as $\dim R_\infty = d + 1 = \dim S_\infty$, Lemma 4.3 implies that

$$\mathcal{P}(\mathcal{M}^*) \cong \text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty) \cong \text{Hom}_{R_\infty}(\mathcal{P}(\mathcal{M}), \omega_{R_\infty})$$

as $R_\infty$-modules (where we have used the fact that $\omega_{S_\infty} = S_\infty$).

We also note that dualizable patching systems are automatically reflexive in the following sense:

**Proposition 4.5.** If $\mathcal{R}$ is a weak patching $\mathcal{R}$-algebra, and $\mathcal{M}$ is a dualizable patching $\mathcal{R}$-module, then the natural map $\mathcal{M} \rightarrow \mathcal{M}^{**}$ given by $x \mapsto (f \mapsto f(x))$ is an isomorphism.

Consequently we have that the map $\mathcal{P}(\mathcal{M}) \rightarrow \text{Hom}_{S_\infty}(\text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty), S_\infty)$ is an isomorphism, and if $R_\infty$ is a cover for $\mathcal{R}$ then $\mathcal{P}(\mathcal{M}) \rightarrow \text{Hom}_{R_\infty}(\text{Hom}_{R_\infty}(\mathcal{P}(\mathcal{M}), \omega_{R_\infty}), \omega_{R_\infty})$ is an isomorphism.

**Proof.** As each $M_n$ is free of finite rank over $S_\infty/\text{Ann}_{S_\infty}(M_n)$, it is a reflexive $S_\infty/\text{Ann}_{S_\infty}(M_n)$-module, and so the first claim follows.

It thus follows that the map $\mathcal{M} \rightarrow \mathcal{M}^{**}$ induces an isomorphism $\mathcal{P}(\mathcal{M}) \sim \rightarrow \mathcal{P}(\mathcal{M}^{**})$. But now by Theorem 4.4 we have natural isomorphisms:

$$\text{Hom}_{S_\infty}(\text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), S_\infty), S_\infty) \cong \text{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}^*), S_\infty) \cong \mathcal{P}(\mathcal{M}^{**})$$

$$\text{Hom}_{R_\infty}(\text{Hom}_{R_\infty}(\mathcal{P}(\mathcal{M}), \omega_{R_\infty}), \omega_{R_\infty}) \cong \text{Hom}_{R_\infty}(\mathcal{P}(\mathcal{M}^*), \omega_{R_\infty}) \cong \mathcal{P}(\mathcal{M}^{**})$$
which clearly identify the maps in the statement of the Proposition with the isomorphism $\mathcal{P}(\mathcal{M}) \sim \mathcal{P}(\mathcal{M}^{**})$.

\[
\begin{align*}
\text{References} & \\
\text{[Eis95]} & \text{David Eisenbud, Commutative algebra, Graduate Texts in Mathematics, vol. 150,}
\text{Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960} \\
\text{[Sch15]} & \text{P. Scholze, On the p-adic cohomology of the Lubin-Tate tower, ArXiv e-prints (2015).} \\
\text{[Sta18]} & \text{The Stacks Project Authors, Stacks Project, http://stacks.math.columbia.edu, 2018.}
\end{align*}
\]