Research Statement

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My research interests are in analysis and geometry. More specifically, my recent work has been using analytical tools to extend geometric notions to nonsmooth settings.

1 Background

Classical geometry has led to the development of many important techniques – from the pioneering complex analysis work of Ahlfors to the modulus developments by Beurling and Fuglede to the rigidity theorem of Mostow, smooth geometry has been studied extensively. More recently, there has been a desire to generalize results of this nature to a nonsmooth setting. Works such as [He] and [HKST] further this goal; both work in metric measure spaces with few assumed regularity conditions. Other propositions can be viewed in this metric context; for instance, Cannon’s conjecture on the boundaries of hyperbolic groups can be rephrased in terms of quasisymmetric equivalence of certain topological 2-spheres with the standard $S^2$. Quasisymmetries can be thought of as metric space generalizations of quasiconformal maps (which are themselves generalizations of conformal maps) that distort relative shapes by a controlled amount.

My work fits into this framework; by using hyperbolic fillings I was able to find two extensions of modulus to suitably nice metric spaces that are quasisymmetrically invariant. Doing so has led to another proof of a result in [Ty], namely that a quasisymmetry between two Ahlfors $Q$-regular metric spaces preserves the $Q$-modulus of path families up to a multiplicative constant depending only on the quasisymmetry. Here one should think of an Ahlfors $Q$-regular metric space as a metric space having dimension $Q$ and the $Q$-modulus as an outer measure on curves which behaves well under conformal transformations.

Chiefly important in these extensions is the construction of hyperbolic fillings which is similar to those in [BP] and [BuS]. Given a connected, compact metric space, one may construct a hyperbolic filling for this space which is a graph that encodes the combinatorial geometry of the space at many discrete levels. Hyperbolic fillings serve as hyperbolic interiors with the original space as a boundary “at infinity” – this is analogous to the way the circle $S^1$ can be identified as the boundary of the unit disk model of hyperbolic 2-space $H^2$. The main idea behind the extensions is to use curves in the filling instead of curves on the boundary when defining modulus. Because these hyperbolic fillings behave well under quasisymmetric maps on the boundary spaces, the quasisymmetric invariance discussed above is straightforward.

2 Details

In what follows we assume all metric measure spaces are compact, connected, Ahlfors $Q$-regular, and have diameter bounded above by 1. The regularity condition says that the measure of any ball
of radius \( r \leq 1 \) is, up to a fixed multiplicative constant, \( r^Q \). The diameter condition can be achieved for any compact metric space by rescaling the metric. Measure metric spaces will be denoted \( Z \) and \( W \), while hyperbolic fillings will be denoted \( X \) and \( Y \).

As mentioned above, one of the primary notions of equivalence in nonsmooth geometry is that of quasisymmetric equivalence. A homeomorphism \( \varphi : Z \to W \) between two metric spaces is called a quasisymmetry if there is a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that for all \( a, b, c \in Z \),

\[
|a - b| \leq t|a - c| \implies |\varphi(a) - \varphi(b)| \leq \eta(t)|\varphi(a) - \varphi(c)|.
\]

The intuition here is that quasisymmetries distort relative shapes by a controlled amount (much like quasiconformal maps in the classical setting). One can show that if \( \varphi \) is a quasisymmetry then \( \varphi^{-1} \) is as well. Thus, metric spaces are partitioned into equivalence classes with two spaces being equivalent if there is a quasisymmetric map between them. Classifying these equivalence classes still poses challenges and, as noted above, is desirable for proving results such as Cannon’s conjecture.

Hyperbolic fillings are graphs associated to metric spaces which transform nicely when a quasi-isometry. This fact forms the foundation for the quasisymmetric invariance of the modulus generalizations in my research.

There are two main approaches to extending modulus in my research. Both of these are modelled on the definition of modulus; instead of paths in the original space, however, paths in the hyperbolic filling are used. Traditional modulus is defined as follows: given a family \( \Gamma \) of rectifiable paths, a function \( \rho : Z \to [0, \infty] \) is admissible for \( \Gamma \) if \( \int_{\gamma} \rho \geq 1 \) (where all path integrals are with respect to the arclength parameterization). The \( p \)-modulus of the path family \( \Gamma \) is then inf \( \int_Z \rho^p \) where the infimum is taken over all admissible \( \rho \).

In the first generalization, \( \text{wcap}_p \), only the modulus of paths families connecting two open sets with positive separation is defined. Let \( A \) and \( B \) be two open sets with dist\((A, B) > 0 \). For admissibility, we consider functions defined on edges of a given hyperbolic filling \( X \) and infinite paths with limits in \( A \) and \( B \). We call such a function \( \tau \) admissible if \( \sum \tau(e) \geq 1 \) for all of these infinite paths. We then define the \( (\text{weak}) \) \( p \)-capacity of \( A \) and \( B \) as \( \text{wcap}_p(A, B) = \inf ||\tau||_{p,\infty} \) where the infimum is taken off all admissible \( \tau \) and \( ||\cdot||_{p,\infty} \) denotes the weak \( \ell^p \)-norm. Recall the weak norm of a function \( f : X \to \mathbb{C} \) defined on a countable space \( X \) is the infimum of \( C \) such that

\[
\# \{ x : |f(x)| > \lambda \} \leq \frac{C \rho}{\lambda^p}.
\]

for all \( \lambda > 0 \). Strictly speaking this is not a norm, but it is comparable to one provided \( p > 1 \). The use of the weak norm here and not the usual \( \ell^p \) norm is motivated by [BoS].

The second generalization relies on approximating paths on the boundary with paths in the hyperbolic filling. For a given finite path \( \gamma : [0, 1] \to Z \), we define a projection onto a suitable subset of a hyperbolic filling \( X \). For a given \( \tau \) defined on the edges of \( X \), we define the \( \tau \)-length of such a projection by summing over the edges in that projection. By expanding the suitable subsets to higher and higher levels (intuitively those which better approximate \( Z \)) and infimizing we arrive
at a similar criteria for admissibility. The mass of $\tau$ is as with $\text{wc}\text{-}cap_p$; that is, $\inf \|\tau\|_{p,\infty}^p$. This extension is denoted $\text{wc}\text{-}cap_p$ for (weak) covering $p$-capacity.

In [Li] it is shown that both of these extensions have the desired quasisymmetric invariance. The more difficult part comes in showing these actually are extensions – that is, that these quantities are comparable to the standard $p$-modulus under the appropriate conditions. One thing to note is that $\text{wc}\text{-}cap_p$ doesn’t rely on rectifiable paths whereas modulus does. For this reason we should expect that in general there is a constant $C$ such that for all open $A, B$ with $\text{dist}(A, B) > 0$ we have $\text{mod}_p(A, B) \leq C \text{wc}\text{-}cap_p(A, B)$ (here $\text{mod}_p(A, B)$ denotes the $p$-modulus of the path family connecting $A$ to $B$). For the other direction one needs a way to guarantee the existence of many rectifiable curves. For this, the Loewner condition is used.

**Theorem.** Let $Q > 1$ and let $(Z,d,\mu)$ be a compact, connected Ahlfors $Q$-regular metric space which is also a $Q$-Loewner space. Let $A, B \subseteq Z$ be open sets with $\text{dist}(A, B) > 0$. Then, there exist constants $c, C > 0$, depending only on $Q$ and the hyperbolic filling parameters, such that

$$c \text{mod}_Q(A, B) \leq \text{wc}\text{-}cap_Q(A, B) \leq C \text{mod}_Q(A, B).$$

Note $Q$ is used here in place of $p$ as it is important to match the Ahlfors regularity dimension.

The second quantity, $\text{wc}\text{-}cap_p$, is also comparable to modulus with appropriate $p$. In this case, however, one does not need to impose the Loewner condition.

**Theorem.** Let $Q > 1$ and let $(Z,d,\mu)$ be a compact, connected Ahlfors $Q$-regular metric space. Then, there exist constants $c, C > 0$ depending only on $Q$ and the hyperbolic filling parameters such that for all path families $\Gamma$,

$$c \text{mod}_Q(\Gamma) \leq \text{wc}\text{-}cap_Q(\Gamma) \leq C \text{mod}_Q(\Gamma).$$

The proofs of these comparability results follow the same scheme: from an admissible function for the presumed larger quantity construct an admissible function for the smaller quantity. Showing the mass of the constructed function is controlled by the mass of the original function then shows the result. The rough ideas behind these two extensions are now summarized. To go from a function $\rho$ on $Z$ to a function $\tau$ on the edges of $X$, we first map to vertices $v$ by $\tau'(v) = r(B_v)\int_{B_v} \rho$ where $B_v$ is the ball that defined the vertex $v$ in the construction, $r(B_v)$ is the radius of this ball, and $\int$ denotes the average integral. Then to map this to an edge $e$ simply add the two $\tau'$ values of the vertices of $e$. To go from a function $\tau$ defined on vertices of $X$ to $Z$, one needs to come up with a method to “push” the values onto $Z$ through some limiting procedure. More precisely, one can use the values of $\tau$ on the balls of a given level to generate functions

$$u_n = \sum_B \frac{\tau(B)}{r(B)} \chi_{2B}$$

where the sum is over the balls corresponding to vertices on level $n$. One then shows that a suitable subsequence of the $u_n$ converges to an admissible $\rho$ (after some modifications).

While the proof strategy is straightforward, some of the details rely on some difficult results. In particular, the fact that the Poincaré inequality is an open condition as proven in [KZ] is used for the Loewner direction and a maximal function estimate due to [BoS] is used for the mass bounds of the subsequential $u_n$ limit.
3 Future Work

With these results in hand, there are a number of options for future developments. One direction of interest is to expand the definition of \( \text{wcap}_p(A, B) \) to pairs of sets more general than open sets with positive separation. The primary difficulty here is that in the proof of comparability between \( \text{wcap}_Q \) and \( \text{mod}_Q \), it was desirable to find “short paths” from the interior of the hyperbolic filling to the boundary sets \( A \) and \( B \). Without an openness condition on \( A \) or \( B \), one would need a stronger result asserting the existence of such short paths to other boundary sets. One can of course define \( \text{wcap}_p(E, F) \) for arbitrary sets with positive separation by taking infimums over pairs of positively separated open covers, but a short path result does not seem out of the question and would avoid this additional infimum. In particular, it would be ideal to define \( \text{wcap}_p(E, F) \) between disjoint, nontrivial continua (here a continuum is a compact, connected set and a nontrivial continuum is one with more than one point).

Another direction of interest is to apply the developed modulus generalizations in contexts where similar structures have succeeded. For instance, in [BK] the structures of \( K \)-approximations are used to formulate a necessary and sufficient condition for particular metric spaces to be quasisymmetrically equivalent to \( S^2 \). In the \( \text{wcap}_p \) setting, it may be possible to use the hyperbolic filling in place of the \( K \) approximations to prove a similar condition. In [BouK] the Combinatorial Loewner Property (CLP) is discussed. It is known that a Loewner space always satisfies the CLP and it would be interesting to find where a capacity Loewner property might fit into this hierarchy when defined analogously.

The quasisymmetric invariance of the quantities \( \text{wcap}_p \) and \( \text{wc-cap}_p \) illustrate that they may be useful in many other contexts in which quasisymmetric maps abound. This becomes especially compelling when the original settings involve two spaces, one of which acting as a boundary for the other. From this one may speculate connections with Cannon’s conjecture or expanding Thurston maps (see, for instance, Theorem 1.9 in [BM]).

References


