How to Find Bases for Jordan Canonical Forms Version 2.0

Unfortunately the method in the previous Jordan canonical form basis pdf doesn’t neatly overlap the method being used in class. To this end, I decided to redo this pdf using that (or something close to that) method.

The Difference

We’ll start as before:

Recall for a linear transformation \( T : V \to V \), we computed a matrix \( A \) for \( T \) and found the characteristic polynomial \( p_A(x) = \prod_{k=1}^{m}(x - \lambda_k)^{\alpha_k} \) with \( \lambda_k \neq \lambda_j \) for \( k \neq j \) and \( \alpha_k \geq 1 \) for all \( k \). Then, for each \( i \), set \( d_i = \dim N((A - \lambda_k I)^i) \) (\( d_0 = 0 \)) from which we found the Jordan canonical form.

The idea for finding a basis relates to the proof of why a Jordan canonical form exists. What we seek to do is find a largest possible set of chains (or cycles) of the form \( \{ x, (T - \lambda_k I)(x), \ldots, (T - \lambda_k I)^{r-1}(x) \} \) which are linearly independent. By the proof of Jordan canonical form, the number and lengths of these chains can be found from the numbers \( d_0, \ldots, d_k \). Instead of looking at the ranges of powers of \( (T - \lambda_k I)|_{W_{\lambda_k}} \) (where \( W_{\lambda_k} \) is the generalized eigenspace corresponding to \( \lambda_k \)), we can find these chains from using quotients of the null spaces.

To begin with, for a fixed eigenvalue \( \lambda \) let \( V_0 = \{ 0 \} \) and \( V_j = N((T - \lambda I)^j) \). Then, we have an increasing sequence \( V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r = N((T - \lambda I)^r) \) (say \( r \) is the first index for which we get this entire nullspace). By examining \( V_{i+1}/V_i \) for \( 0 \leq i \leq r - 1 \), we will construct chains as before. First, consider \( V_r/V_{r-1} \). This is a vector space of dimension \( \geq 1 \) (as \( r \) was the first index at which we got all of \( N((T - \lambda I)^r) \), so it has a basis, say \( \{ z_1 + V_{r-1}, \ldots, z_{m_r} + V_{r-1} \} \). Then, \( \{ z_1, \ldots, z_{m_r} \} \) is a linearly independent set in \( V_r \) for which \( (T - \lambda I)^r(z_i) = 0 \) and \( (T - \lambda I)^{r-1}(z_i) \neq 0 \) for each \( 1 \leq i \leq m_r \). Hence, we get the chains \( \{ z_i, (T - \lambda I)(z_i), \ldots, (T - \lambda I)^{r-1}(z_i) \} \). Now, consider \( V_{r-1}/V_{r-2} \). We already have the vectors \( \{ (T - \lambda I)(z_1) + V_{r-2}, \ldots, (T - \lambda I)(z_{m_r}) + V_{r-2} \} \) forming a linearly independent set in this vector space. If \( \dim(V_{r-1}/V_{r-2}) = m_{r-1} \), then we move on to \( V_{r-2}/V_{r-3} \). If not, then we can extend this to a basis by adding \( \{ y_1 + V_{r-2}, \ldots, y_{m_{r-1}} + V_{r-2} \} \). Then, we get the chains \( \{ y_1, (T - \lambda I)(y_1), \ldots, (T - \lambda I)^{r-2}(y_1) \} \) which add to our previous chains. After this, we move on to \( V_{r-2}/V_{r-3} \) with the linearly independent set \( \{ (T - \lambda I)^2(z_1) + V_{r-3}, \ldots, (T - \lambda I)^2(z_{m_r}) + V_{r-3}, (T - \lambda I)(y_1) + V_{r-3}, \ldots, (T - \lambda I)(y_{m_{r-1}}) + V_{r-3} \} \). We repeat the extension process above if necessary and continue until we have done this through \( V_1/V_0 \). Then, the union of all the chains we have found will be our Jordan canonical basis for \( W_{\lambda} \). Taking the union of each of these bases gives a Jordan canonical basis for the whole space.

Rough Step-by-Step Outline of the Method

(i) Let \( A \) be a matrix for \( T \) and find \( p_A(x) \) and factor it as before. For each eigenvalue \( \lambda \), find the \( d_i \)’s.
(ii) Compute each \( V_j = N((T - \lambda I)^j) \) until \( d_j = \dim V_j = \alpha \) (the multiplicity of the chosen \( \lambda \))
(iii) Find the chains by looking for bases of the various quotient spaces that have been extended from previously found independent vectors. For this, start at \( V_r/V_{r-1} \) (with \( r \) as above) and stop after \( V_1/V_0 \). Then, take the union of each of these chains to form a basis \( \beta_\lambda \) for the generalized eigenspace \( W_{\lambda} \).
(iv) Applying (iii) for each \( \lambda \) will give a basis \( \beta_\lambda \) for each generalized eigenspace \( W_{\lambda} \). Combining these bases will give the desired Jordan canonical form basis (up to rearrangement).

Examples

Let’s find Jordan canonical form bases for the examples from last time.

Example 1. Let \( T : V \to V \) be a transformation with matrix \( A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix} \).
Recall $p_A(x) = (x-1)^3$. We had $A-I = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ which has rank 1 and $(A-I)^2 = 0$. Thus, $d_0 = 0$, $d_1 = 2$, and $d_2 = 3$. As $N((A-I)^2) = V$, we need only compute $N((A-I))$. Now (using the convention as before of writing a column vector as $(a,b,c)$ for typesetting purposes, $(A-I)(a,b,c) = (a+2b+3c, a+2b+3c, -a-2b-3c)$, so $N((A-I)) = \{(a,b,c)|a+2b+3c=0\}$. Hence, we have $V_0 = \{0\}$, $V_1 = \{(a,b,c)|a+2b+3c=0\}$, and $V_2 = V$. Thus, $V_2/V_1$ has dimension $d_2 - d_1 = 3 - 2 = 1$. To find a nonzero vector in $V_2/V_1$, all we need to do is find an $(a,b,c)$ with $a+2b+3c \neq 0$. For this, we may take $a = 1$, $b = c = 0$. Then, we get the chain $\{(1,0,0), (A-I)(1,0,0)\} = \{(1,0,0), (1,1,-1)\}$.

Now, in $V_1/V_0$ (which can be thought of as just $V_1$ as $V_0 = \{0\}$), we already have the vector $(1,1,-1)+V_0$. As $\dim(V_1/V_0) = d_1 - d_0 = 2 - 0 = 2$, we can extend $\{(1,1,-1) + V_0\}$ to a basis of $V_1/V_0$ by finding a vector $(a,b,c) \in V_1$ such that $\{(1,1,-1) + V_0, (a,b,c)+V_0\}$ is linearly independent. That is, we need to find $(a,b,c)$ such that $\{(1,1,-1), (a,b,c)\}$ is a basis for $N((A-I))$. We must have $a+2b+3c = 0$ as above, so $(-2,1,0)$ works. This gives us the chain $\{(-2,1,0)\}$ which, when added to our other chain, gives us the basis $\beta_1 = \{(1,0,0), (1,1,-1), (-2,1,0)\}$.

As 1 is the only eigenvalue of $A$, this is a Jordan canonical form basis for $T$. Indeed, we see $[T]_{\beta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (if we wish to have $[1 & 0 & 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ from last time, we need only reorder the chains and the elements within the chains; if $\beta' = \{(-2,1,0), (1,1,-1), (1,0,0)\}$, then $[T]_{\beta'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$.

**Example 2.** Let $T : V \rightarrow V$ be a transformation with matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$. We recall $p_A(x) = x^4(x-1)$. We had

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

with rank 3,

$$A - \lambda I)^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

with rank 2, and

$$A - \lambda I)^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

with rank 1. Hence, for the eigenvalue $\lambda = 0$, we have $d_0 = 0$, $d_1 = 2$, $d_2 = 3$, $d_3 = 4$. We see $A(a,b,c,d,e) = (a,a-b-e,a-b-e,-e,-a+b+e)$ so $V_1 = N(A) = \{(a,b,c,d,e)|a=0, e=0, and b=0\}$. Likewise, $A^2(a,b,c,d,e) = (a,a,a-a-b-e,-a)$ so $V_2 = N(A^2) = \{(a,b,c,d,e)|a=0 and b=-e\}$. We have $A^3(a,b,c,d,e) = (a,a,a,a,-a)$ so $V_3 = N(A^3) = \{(a,b,c,d,e)|a=0\}$. As $d_3 = \alpha_0$, we stop here.
Now, $V_2/V_2$ has dimension $d_3 - d_2 = 1$, so to find a basis we need only find one vector in $V_2 \setminus V_2$. That is, we need to find $(a, b, c, d, e)$ with $a = 0$ but $b \neq -e$. For this, $(0, 1, 0, 0, 1)$ will do. From this vector we get the chain

$$\{(0, 1, 0, 0, 1), A(0, 1, 0, 0, 1), A^2(0, 1, 0, 0, 1)\} = \{(0, 1, 0, 0, 1), (0, -2, -2, -1, 2), (0, 0, 0, -2, 0)\}$$

Next we look at $V_2/V_1$. Here we already have the linearly independent set $\{(0, 0, 0, -2, 0) + V_0\}$ from our first chain. As $\dim(V_2/V_1) = d_2 - d_1 = 1$, there are no more basis vectors to be found. Hence, we move on to $V_1/V_0$.

For $V_1/V_0$, we already have the linearly independent set $\{(0, 0, 0, -2, 0) + V_0\}$ from our first chain. We see $\dim(V_1/V_0) = d_1 - d_0 = 2$, so there is another basis vector to be found. That is, we seek $(a, b, c, d, e) \in V_1$ such that $\{(0, 0, 0, -2, 0), (a, b, c, d, e)\}$ is linearly independent. As $(a, b, c, d, e) \in V_1$, we must have $a = 0$, $b = 0$, and $e = 0$. Thus, we can take $(0, 0, 1, 0, 0)$. This gives us the chain $\{(0, 0, 1, 0, 0)\}$ which, when added to the previous chain, produces our basis $\beta_0 = \{(0, 1, 0, 0, 1), (0, -2, -2, -1, 2), (0, 0, 0, -2, 0), (0, 0, 1, 0, 0)\}$ for $V_0$.

We also need a Jordan basis for $W_1$, but this is easy as $\dim W_1 = 1$. Thus, we only need to find $(a, b, c, d, e) \in N(A - I)$. We see $A - I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}$ and so $(A - I)(a, b, c, d, e) = (0, a - 2b - e, a - b - c - e, -d - e, -a + b)$. The only solution for this with $a = 0$ is $(0, 0, 0, 0, 0)$, so we may take $a = 1$. Then, this gives $b = 1$, $e = -1$, $c = 1$, $d = 1$. Thus, $(1, 1, 1, 1, -1)$ is our desired vector, so $\beta_1 = \{(1, 1, 1, 1, -1)\}$ is a basis for $W_1$. Hence,

$$\beta = \beta_1 \cup \beta_0 = \{(1, 1, 1, 1, -1), (0, 1, 0, 0, 1), (0, -2, -2, -1, 2), (0, 0, 0, -2, 0), (0, 0, 1, 0, 0)\}$$

is a Jordan canonical form basis for $T$. Indeed,

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Example 3.** Let $T : V \rightarrow V$ be a transformation with matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Recall $p_A(x) = (x - 1)^6$. We have

$$A - I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
with rank 3,

\[
(A - I)^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

with rank 1, and \((A - I)^3 = 0\). Thus, for \(\lambda = 1\), \(d_0 = 0\), \(d_1 = 3\), \(d_2 = 5\), and \(d_3 = 6\). We see \((A - I)(a, b, c, d, e, f) = (0, -b - e + f, -a - b + d - e + f, 0, b + e, 0)\) so we have \(V_1 = N((A - I)) = \{(a, b, c, d, e, f) \mid b + e = f, -a - b + d - e + f = 0, \text{ and } b + e = 0\}\) which is \(V_1 = \{(a, b, c, d, e, f) \mid b + e = 0, a = d, \text{ and } f = 0\}\). Now, \((A - I)^2(a, b, c, d, e, f) = (0, -f, -f, 0, f, 0)\) so \(V_2 = N((A - I)^2) = \{(a, b, c, d, e, f) \mid f = 0\}\). Lastly, \(V_3 = V\).

Now, \(V_3/V_2\) has dimension \(d_3 - d_2 = 1\), so we need only find one vector in \(V_3\setminus V_2\). It is clear that \((0, 0, 0, 0, 0, 1)\) will work. Thus, we get the chain

\[
\{(0, 0, 0, 0, 0, 1), (A - I)(0, 0, 0, 0, 0, 1), (A - I)^2(0, 0, 0, 0, 0, 1)\} = \{(0, 0, 0, 0, 0, 1), (0, 1, 1, 0, 0, 0), (0, -1, -1, 0, 1, 0)\}
\]

Next we look at \(V_2/V_1\). This has dimension \(d_2 - d_1 = 5 - 3 = 2\). From our first chain, we have the linearly independent set \(\{(0, 1, 1, 0, 0, 0) + V_1\}\). Thus, we seek to find a vector \((a, b, c, d, e, f) \in V_2\setminus V_1\) such that \(\{(0, 1, 1, 0, 0, 0) + V_1, (a, b, c, d, e, f) + V_1\}\) is linearly independent. As \((a, b, c, d, e, f) \in V_2\), we have \(f = 0\). To find \((a, b, c, d, e, 0)\), we choose a basis of \(V_1\), say \(\gamma = \{(1, 0, 0, 1, 0, 0), (0, 1, 0, 0, -1, 0), (0, 0, 1, 0, 0, 0)\}\) and extend it to a basis of \(V_2\) including \((0, 1, 1, 0, 0, 0)\). Thus, we wish to find \((a, b, c, d, e, 0)\) such that

\[
\{(a, b, c, d, e, 0), (1, 0, 0, 1, 0, 0), (0, 1, 0, 0, -1, 0), (0, 0, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0)\}
\]

is linearly independent. As only \((1, 0, 0, 1, 0, 0)\) has nonzero 1\(^{st}\) and 4\(^{th}\) coefficients, we see \((1, 0, 0, 0, 0, 0)\) works. Thus, \(\{(0, 1, 1, 0, 0, 0) + V_1, (1, 0, 0, 0, 0, 0) + V_1\}\) is our basis for \(V_2/V_1\) which gives us the chain

\[
\{(1, 0, 0, 0, 0, 0), (A - I)(1, 0, 0, 0, 0, 0)\} = \{(1, 0, 0, 0, 0, 0), (0, 0, -1, 0, 0, 0)\}.
\]

Lastly we look at \(V_1/V_0\). This as dimension \(d_1 - d_0 = 3\). From our previous chains, we have the linearly independent set \(\{(0, -1, -1, 0, 1, 0) + V_0, (0, 0, -1, 0, 0, 0) + V_0, (a, b, c, d, e, f) + V_0\}\). Hence, we are missing a vector. We wish to find \((a, b, c, d, e, f) \in V_1\) such that \(\{(0, -1, -1, 0, 1, 0) + V_0, (0, 0, -1, 0, 0, 0) + V_0, (a, b, c, d, e, f) + V_0\}\) is linearly independent. As \((a, b, c, d, e, f) \in V_1\), we must have \(b + e = 0, a = d, \text{ and } f = 0\). Thus, \((a, b, c, d, e, f) = (a, b, c, a, -b, 0)\) and \(\{(0, -1, -1, 0, 1, 0), (0, 0, -1, 0, 0, 0), (a, b, c, a, -b, 0)\}\) is linearly independent. Using \((0, -1, -1, 0, 1, 0) - (0, 0, 0, -1, 0, 0) = (0, -1, 0, 0, 1, 0)\) we can zero out the 2\(^{nd}\) and 5\(^{th}\) coefficients and using \((0, 0, -1, 0, 0, 0)\) we can zero out the 3\(^{rd}\) coefficient, so \((1, 0, 0, 1, 0, 0)\) will work. This gives us the chain \(\{(1, 0, 0, 1, 0, 0)\}\).

Adding this to our previous chains, we get the basis

\[
\beta = \{(1, 0, 0, 1, 0, 0), (1, 0, 0, 0, 0, 0), (0, 0, -1, 0, 0, 0), (0, 0, 0, 0, 0, 1), (0, 1, 1, 0, 0, 0), (0, -1, -1, 0, 1, 0)\}
\]

our Jordan canonical basis. In this basis,

\[
[T]_\beta = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]