SEQUENCE COMMUTATION PROOFS

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1. Definition

This document defines sequence commuation and shows thats properties can be derived directly from those of pairwise patch commutation. I've separated it out from the rest of the discussion because it's somewhat notation-heavy. However, the choice of notation makes these proofs pretty straightforward.

Sequence commutation can also be defined inductively, as in [1]. That treatment does not directly prove most of the propositions of this document, but I don't think there's any technical reasons for the omission. Informally, it seems clear that the two definitions are equivalent.

Definition 1.1. Let $(\mathcal{P}, S, \mathcal{E})$ be a patch system, and let $\leftrightarrow$ be a binary relation on $\mathcal{P}$. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ be elements of $\mathcal{P}^*$, with

$$\mathbf{p} = (p_0, \dots, p_m),\ \mathbf{q} = (q_0, \dots, q_n),\ \mathbf{r} = (r_0, \dots, r_n),\ \mathbf{s} = (s_0, \dots, s_m).$$

We say that $(\mathbf{p}, \mathbf{q}) \leftrightarrow^* (\mathbf{r}, \mathbf{s})$ if there exist

$$p_i^{(k)} \text{ for } 0 \le i \le m,\ 0 \le k \le n+1,$$
$$q_j^{(l)} \text{ for } 0 \le j \le n,\ 0 \le l \le m+1.$$

such that

$$p_i^{(0)} = p_i,\ q_j^{(0)} = q_j, \tag{1}$$

$$p_i^{(n+1)} = s_i,\ q_j^{(m+1)} = r_j, \text{ and} \tag{2}$$

$$\left(p_{m-i}^{(j)}, q_j^{(i)}\right) \leftrightarrow \left(q_j^{(i+1)}, p_{m-i}^{(j+1)}\right). \tag{3}$$

The following diagram illustrates the relationships between the indices.

$$\begin{tikzcd}
{} \arrow[r, "p_0"] \arrow[d, "q_0^{(3)}"] & {} \arrow[r, "p_1"] \arrow[d, "q_0^{(2)}"] & {} \arrow[r, "p_2"] \arrow[d, "q_0^{(1)}"] & {} \arrow[d, "q_0"] \\
{} \arrow[r, "p_0^{(1)}"] \arrow[d, "q_1^{(3)}"] & {} \arrow[r, "p_1^{(1)}"] \arrow[d, "q_1^{(2)}"] & {} \arrow[r, "p_2^{(1)}"] \arrow[d, "q_1^{(1)}"] & {} \arrow[d, "q_1"] \\
{} \arrow[r, "p_0^{(2)}"] & {} \arrow[r, "p_1^{(2)}"] & {} \arrow[r, "p_2^{(2)}"] & {}
\end{tikzcd}$$

Proposition 1.1. *Let $(\mathcal{P}, S, \mathcal{E})$ be a patch system with commutation relation $\leftrightarrow$. Then Definition 1.1 defines a commutation relation $\leftrightarrow^*$ for the patch system $(\mathcal{P}^*, S, \mathcal{E}^*)$.*

Furthermore, if $\leftrightarrow$ is uniquely commuting, then so is $\leftrightarrow^*$.

2. Proofs of commutation relation proerties

In this section we will show that $\leftrightarrow^*$ is a commutation relation on $\mathcal{P}^*$. Let $(\mathcal{P}, S, \mathcal{E})$ be a patch system with associated commutation relation $\leftrightarrow$.

To show that $\leftrightarrow^*$ is effect-preserving, we need a concrete way to commute two sequences using pairwise patch swaps. We arbitrarily choose to commute each element of q all the way to the left, one at a time. The following lemma shows that the effect of a sequence is preserved at each step:

Lemma 2.1. *Let $p_i^{(j)}$, $q_j^{(i)} \in \mathcal{P}$ satisfy equations (1) and (3). Let*

$$\mathbf{S}_{i,j} = \left(q_0^{(m+1)}, \dots, q_{j-1}^{(m+1)}, p_0^{(j)}, \dots, p_{m-i}^{(j)}, q_j^{(i)}, p_{m-i+1}^{(j+1)}, \dots, p_m^{(j+1)}, q_{j+1}^{(0)}, \dots, q_n^{(0)}\right).$$

Then $\mathcal{E}(\mathbf{S}_{i,j}) = \mathcal{E}(\mathbf{pq})$ for $0 \le i \le m+1$ and $0 \le j \le n+1$.

Proof. Note that $\mathbf{S}_{0,0} = \mathbf{pq}$ and $\mathbf{S}_{m+1,j} = \mathbf{S}_{0,j+1}$. Therefore by induction it suffices to prove that $\mathcal{E}(\mathbf{S}_{i+1,j}) = \mathcal{E}(\mathbf{S}_{i,j})$ for $0 \le i \le m$, $0 \le j \le n+1$.

But we can obtain $\mathbf{S}_{i+1,j}$ from $\mathbf{S}_{i,j}$ by replacing the subsequence $\left(p_{m-i}^{(j)}, q_j^{(i)}\right)$ with $\left(q_j^{(i+1)}, p_{m-i}^{(j+1)}\right)$; so the result follows from (3) since $\leftrightarrow$ is effect-preserving. ☐

Proposition 2.2. *The relation $\leftrightarrow^*$ is effect-preserving.*

Proof. Assume $(\mathbf{p}, \mathbf{q}) \leftrightarrow^* (\mathbf{r}, \mathbf{s})$. Let $\mathbf{S}_{i,j}$ be as in Lemma 2.1. Then $\mathbf{rs} = \mathbf{S}_{m+1,n+1}$ by (2), so $\mathcal{E}(\mathbf{rs}) = \mathcal{E}(\mathbf{S}_{m+1,n+1}) = \mathcal{E}(\mathbf{pq})$. ☐

Proposition 2.3. *The relation $\leftrightarrow^*$ is symmetric.*

Proof. Assume $(\mathbf{p}, \mathbf{q}) \leftrightarrow^* (\mathbf{r}, \mathbf{s})$, and let $p_i^{(k)}$, $q_j^{(l)}$ be as in Definition 1.1. To show that $(\mathbf{r}, \mathbf{s}) \leftrightarrow^* (\mathbf{p}, \mathbf{q})$, we must find some $\tilde{p}_i^{(k)}$ and $\tilde{q}_j^{(l)}$ which also satisfy that definition, but with (1) and (3) swapped.

Let $\tilde{p}_i^{(k)} = p_i^{(n-k+1)}$ and $\tilde{q}_j^{(l)} = q_j^{(m-l+1)}$. Then $\tilde{\mathbf{p}}^{(0)} = \mathbf{p}^{(n+1)} = \mathbf{s}$ and $\tilde{\mathbf{p}}^{(n+1)} = \mathbf{p}^{(0)} = \mathbf{p}$; similarly $\tilde{\mathbf{q}}^{(0)} = \mathbf{r}$ and $\tilde{\mathbf{q}}^{(m+1)} = \mathbf{q}$.

Furthermore, since $\leftrightarrow$ is symmetric,

$$\begin{aligned}
&\left(p_{m-i}^{(j)}, q_j^{(i)}\right) \leftrightarrow \left(q_j^{(i+1)}, p_{m-i}^{(j+1)}\right) \\
\implies &\left(q_j^{(i+1)}, p_{m-i}^{(j+1)}\right) \leftrightarrow \left(p_{m-i}^{(j)}, q_j^{(i)}\right) && \text{for all } i,\ j \\
\implies &\left(q_{n-j}^{(m-i+1)}, p_i^{(n-j+1)}\right) \leftrightarrow \left(p_i^{(n-j)}, q_{n-j}^{(m-i)}\right) && \text{for all } i,\ j \\
\implies &\left(\tilde{q}_{n-j}^{(i)}, \tilde{p}_i^{(j)}\right) \leftrightarrow \left(\tilde{p}_i^{(j+1)}, \tilde{q}_{m-j}^{(i+1)}\right).
\end{aligned}$$

Thus $(\mathbf{r}, \mathbf{s}) \leftrightarrow^* (\mathbf{p}, \mathbf{q})$.

☐

Proposition 2.4. *The relation $\leftrightarrow^*$ is rotating.*

Proof. Assume $(\mathbf{p}, \mathbf{q}) \leftrightarrow^* (\mathbf{r}, \mathbf{s})$, and let $p_i^{(k)}$, $q_j^{(l)}$ be as above. We will use the notation $\hat{q}_i^{(k)} = \left(q_i^{(k)}\right)^{-1}$.

Let $\tilde{q}_i^{(l)} = \hat{q}_{n-j}^{(m-l+1)}$. Then $\tilde{\mathbf{q}}^{(0)} = (\hat{q}_n^{(m+1)}, \dots, \hat{q}_0^{(m+1)}) = \mathbf{r}^{-1}$ and $\tilde{\mathbf{q}}^{(m+1)} = (\hat{q}_n^{(0)}, \dots, \hat{q}_0^{(0)}) = \mathbf{q}^{-1}$.

Furthermore, since $\leftrightarrow$ is rotating,

$$\begin{aligned} &\left(p^{(j)}_{m-i}, q^{(i)}_j\right) \leftrightarrow \left(q^{(i+1)}_j, p^{(j+1)}_{m-i}\right) \\ \implies &\left(\hat{q}^{(i+1)}_j, p^{(j)}_{m-i}\right) \leftrightarrow \left(p^{(j+1)}_{m-i}, \hat{q}^{(i)}_j\right) && \text{for all } i, j \\ \implies &\left(\hat{q}^{(m-i+1)}_j, p^{(j)}_i\right) \leftrightarrow \left(p^{(j+1)}_i, \hat{q}^{(m-i)}_j\right) && \text{for all } i, j \\ \implies &\left(\tilde{q}^{(i)}_{n-j}, p^{(j)}_i\right) \leftrightarrow \left(p^{(j+1)}_i, \tilde{q}^{(i+1)}_{n-j}\right). \end{aligned}$$

Thus $\left(\mathbf{r}^{-1}, \mathbf{p}\right) \leftrightarrow^* \left(\mathbf{s}, \mathbf{q}^{-1}\right)$. □

3. Proof of Uniqueness

Proposition 3.1. *Let* $(\mathcal{P}, S, \mathcal{E})$ *be a patch system, and* $\leftrightarrow$ *be a uniquely commuting relation. Then* $\leftrightarrow^*$ *is also uniquely commuting; i.e., for any* $\mathbf{p}, \mathbf{q} \in \mathcal{P}^*$ *there is at most one pair* $(\mathbf{r}, \mathbf{s})$ *with* $(\mathbf{p}, \mathbf{q}) \leftrightarrow^* (\mathbf{r}, \mathbf{s})$.

Proof. Assume that $(\mathbf{p}, \mathbf{q}) \leftrightarrow^* (\mathbf{r}, \mathbf{s})$ and $(\mathbf{p}, \mathbf{q}) \leftrightarrow^* (\tilde{\mathbf{r}}, \tilde{\mathbf{s}})$. By definition, we have corresponding $p^{(k)}_i, q^{(l)}_j \in \mathcal{P}$ and $\tilde{p}^{(k)}_i, \tilde{q}^{(l)}_j \in \mathcal{P}$ which satisfy Definition 1.1.

We will use induction on $i + j$ to show that

$$p^{(j)}_{m-i} = \tilde{p}^{(j)}_{m-i} \tag{4}$$

$$q^{(i)}_j = \tilde{q}^{(i)}_j. \tag{5}$$

The result then follows from (2).

When $i + j = 0$, we must have $i = j = 0$ so the claims follows trivially from (1). For the inductive step, fix $w > 0$ and assume that (4) and (5) hold for all i and j satisfying $i + j = w - 1$. Then fix i and j with $i + j = w$.

We first consider (4). If $j = 0$, this follows trivially from (1). Otherwise, rewrite (3) as

$$\left(p^{(j-1)}_{m-i}, q^{(i)}_{j-1}\right) \leftrightarrow \left(q^{(i+1)}_{j-1}, p^{(j)}_{m-i}\right).$$

Since $\mathcal{P}$ is uniquely commuting, claim (4) follows from the inductive hypothesis.

Similarly, (5) follows by noting that it is trivial when $i = 0$, and otherwise by rewriting (3) as

$$\left(p^{(j)}_{m-(i-1)}, q^{(i-1)}_j\right) \leftrightarrow \left(q^{(i)}_j, p^{(j+1)}_{m-(i-1)}\right).$$

This completes the inductive step. □

References

[1] Ian Lynagh. Camp patch theory. Available from http://projects.haskell.org/camp/files/theory.pdf, April 2009.

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