This is all taken from Lawson [2]. In what follows, let S be a semigroup.

Definition 1. Let $x \in S$. We say that $y \in S$ is an *inverse* of x if $xyx = x$ and $yxy = y$.

Definition 2. We say that a semigroup S is *regular* if every element has at least one inverse element.

Definition 3. We say a semigroup S is an *inverse semigroup* if every element x has a unique inverse, denoted by x^{-1}.

Definition 4. Let S be a semigroup. We say that $x \in S$ is *idempotent* if $xx = x$.

The following two propositions follow immediately from the definitions.

Proposition 1. *Any idempotent e is its own inverse.*

Proposition 2. *Let y be an inverse of x; then x is an inverse of y.*

Now we can prove the main result:

Theorem 3. *Let S be a regular semigroup. Then S is an inverse semigroup if and only if all idempotents commute; that is, if e and f are idempotents then $ef = fe$.*

Proof. For now, we leave the first half of the proof to Wikipedia [1].

Assume that S is an inverse semigroup, and let e and f be idempotents in S. I claim that the products ef and fe are also idempotent. Then by Prop. 1, $(ef)^{-1} = ef$; however we must also have $(ef)^{-1} = fe$ since

$$ef(fe)ef = efef = ef$$

and

$$fe(ef)fe = fefe = fe.$$

Thus $ef = fe$ by uniqueness of inverses.

It remains to show that the product of any two idempotents e and f is also idempotent. Let $x = (ef)^{-1}$. Then fxe is idempotent, since

$$(fxe)(fxe) = f(xefx)e = fxe.$$

Furthermore fxe is also an inverse of ef, since

$$fxe(ef)fxe = (fxe)(fxe) = fxe$$

and

$$ef(fxe)ef = (ef)x(ef) = ef.$$

So by uniqueness of inverses, $(ef)^{-1} = fxe$ and is idempotent. But by Prop. 2, ef is the inverse of $(ef)^{-1}$, so it must also be idempotent.

□

References

[1] Inverse semigroups. Available from `http://en.wikipedia.org/wiki/Inverse_semigroup`, April 2009.

[2] Mark V. Lawson. *Inverse Semigroups*. World Scientific, New Jersey, 1998.