Exercise 1. Work out in detail the winding number argument to show that there are solutions to \( e^z = z \).

Proof. Let \( f(z) = e^z - z \), and consider a square of side length \( 4\pi N \) centered at 0 with sides parametrized by 
\[
\gamma_1(t) = (1 - 2t)2\pi N + 2\pi iN, \ t \in [0, 1];
\gamma_2(t) = -2\pi N + (1 - 2t)2\pi iN, \ t \in [0, 1];
\gamma_3(t) = (1 + 2t)2\pi N - 2\pi iN, \ t \in [-1, 0];
\gamma_4(t) = 2\pi N + (1 + 2t)2\pi iN, \ t \in [-1, 0].
\]
We will compute the winding number of the image of this square under \( f \).

The image of each side of the square is as follows:
\[
f(\gamma_1(t)) = (e^{(1-2t)2\pi N} - (1 - 2t)2\pi N) - 2\pi iN, \ t \in [0, 1];
\]
\[
f(\gamma_2(t)) = e^{-2\pi N}e^{(1-2t)2\pi iN} + (2\pi N - (1 - 2t)2\pi iN), \ t \in [0, 1];
\]
\[
f(\gamma_3(t)) = (e^{(1+2t)2\pi N} - (1 + 2t)2\pi N) + 2\pi iN, \ t \in [-1, 0];
\]
\[
f(\gamma_4(t)) = e^{2\pi N}e^{(1+2t)2\pi iN} - (2\pi N + (1 + 2t)2\pi iN), \ t \in [-1, 0].
\]
Note that \( f(\gamma_1(t)) \) and \( f(\gamma_3(t)) \) have fixed imaginary parts of \(-2\pi N \) and \( 2\pi N \) and are contained in the fourth and first quadrants, respectively. Note also that \( f(\gamma_2(t)) \) is approximately a vertical line segment connecting \( f(\gamma_1(1)) \) and \( f(\gamma_3(-1)) \) (if \( N \) is large, then \( e^{-2\pi N} \) is very small). Finally, note that \( f(\gamma_4(t)) \) is approximately a circle of radius \( e^{2\pi N} \) that winds around the origin \( 2N \) times, beginning at \( f(\gamma_1(0)) \) and ending at \( f(\gamma_3(0)) \). Thus, we have the picture displayed below:
It follows that $e^z - z$ has $2N$ zeros inside our square. In particular, there are solutions to $e^z = z$.

\[\square\]

**Exercise 2.** The quadratic formula shows that the two roots of $z^2 + 4z + 15 = 0$ lie in the second and third quadrant. Show how this works with winding numbers.

**Proof.** Let $f(z) = z^2 + 4z + 15$, and consider a pie slice of radius $R$ in the second quadrant parametrized by $\gamma_1(t) = it$, $t \in [0, R]$; $\gamma_2(t) = Re^{it}$, $t \in \left[\frac{\pi}{2}, \pi\right]$; and $\gamma_3(t) = t$, $t \in [-R, 0]$. We will compute the winding number of the image of this pie slice under $f$.

First, note that

$$f(\gamma_1(t)) = (15 - t^2) + 4it,$$

so the image of $\gamma_1$ under $f$ is a parabolic arc in the upper-half plane (except for $f(\gamma_1(0)) = 15$). Thus, the argument of $f(\gamma_1(t))$ increases from $0$ to approximately $\pi$.

Next,

$$f(\gamma_2(t)) = R^2 e^{2it} + 4Re^{it} + 15 = R^2 e^{2it} \left(1 + \frac{4}{Re^{it}} + \frac{15}{R^2 e^{2it}}\right),$$

which, for large $R$, is approximately a half-circle of radius $R^2$. Thus, the argument of $f(\gamma_2(t))$ increases from approximately $\pi$ to exactly $2\pi$.

Finally, we see that

$$f(\gamma_3(t)) = t^2 + 4t + 15 \in \mathbb{R} \quad \text{for all} \quad t \in [-R, 0],$$

so the argument of $f(\gamma_3(t))$ is constant.

It follows that the image of the pie slice has winding number $1$ around $z = 0$ (see the image above), and thus that $z^2 + 4z + 15$ has a root in the second quadrant. The computation for the third quadrant is analogous. 

\[\square\]
Exercise 3. Suppose $h$ is harmonic (real-valued) on an open set containing $\{z | |z| \leq 1\}$ and that $h$ is positive on $\{z | |z| = 1\}$ (and hence on $\{z | |z| < 1\}$). Show that there is an $M > 1$ such that

$$M|h(0)| \geq |h(z)| \geq \frac{1}{M}|h(0)|$$

for every $z$ with $0 \leq |z| \leq \frac{1}{2}$. What is the smallest $M$ you can show will work?

Proof. By the Poisson integral formula, we have

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) \frac{1 - r^2}{r^2 - 2r \cos(\theta - t) + 1} dt.$$

We can bound the Poisson kernel for $0 \leq r \leq \frac{1}{2}$ as follows:

$$\frac{1 - r}{1 + r} = \frac{1 - r^2}{1 + 2r + 1} \leq \frac{1 - r^2}{r^2 - 2r \cos(\theta - t) + 1} \leq \frac{1 - r^2}{r^2 - 2r + 1} = \frac{1 + r}{1 - r},$$

so the bounds on $r$ imply that

$$\frac{1}{3} \leq \frac{1 - r}{1 + r} \leq \frac{1 - r^2}{r^2 - 2r \cos(\theta - t) + 1} \leq \frac{1 + r}{1 - r} \leq 3.$$

Hence, we have

$$\frac{1}{3} h(0) = \frac{1}{6\pi} \int_0^{2\pi} h(e^{it}) \frac{1 - r^2}{r^2 - 2r \cos(\theta - t) + 1} dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) \frac{1 - r^2}{r^2 - 2r \cos(\theta - t) + 1} dt$$

$$\leq \frac{3}{2\pi} \int_0^{2\pi} h(e^{it}) dt$$

$$= 3h(0),$$

where the equalities follow from the mean value property of harmonic functions. The result follows. \qed

Exercise 4. Suppose $f$ is holomorphic on $\{z | |z| < 1\}$ and $|f(z)| < 1$ for all $z$ with $|z| < 1$. Suppose also that $f(0) = 0$. Show that $|f(z)| \leq |z|$ for every $z$ with $|z| < 1$.

Proof. Note that, since $\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = f'(0)$, the function $\frac{f(z)}{z}$ has a removable singularity at $z = 0$. Thus, the function

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0, \\ f'(0) & z = 0 \end{cases}$$

is holomorphic in the unit disk.
Now, for \( z \) in the circle of radius \( R < 1 \) centered at 0, we have
\[
\left| \frac{f(z)}{z} \right| = \frac{|f(z)|}{R} < \frac{1}{R}.
\]

Thus, by the maximum modulus principle (applied to \( g \)), \( |f(z)| \leq \frac{|z|}{R} \) for all \( z \) with \( 0 < |z| < R \). If \( R < R' < 1 \), then \( |f(z)| < \frac{|z|}{R'} < \frac{|z|}{R} \) for all \( z \) with \( 0 < |z| < R' \), which includes all \( z \) with \( 0 < |z| < R \). Thus, if we let \( R \to 1 \), we see that \( |f(z)| \leq |z| \) if \( 0 < |z| < 1 \). Since \( f(0) = 0 \), we are done. \( \square \)

**Exercise 5.** What happens in Exercise 4 if \( |f(z)| = |z| \) for some \( z \) with \( 0 < |z| < 1 \)?

**Proof.** In Exercise 4, we proved that \( \left| \frac{f(z)}{z} \right| \leq 1 \) for all \( z \) with \( 0 < |z| < 1 \). By the maximum modulus principle (applied to the function \( g \) from Exercise 4), if \( \left| \frac{f(z)}{z} \right| \) attains a maximum value (in this case, 1) at some \( z \) with \( 0 < |z| < 1 \), then \( \left| \frac{f(z)}{z} \right| = 1 \) for all \( z \) with \( 0 < |z| < 1 \). But, as we showed in Exercise 4,
\[
g(z) = \begin{cases} 
\frac{f(z)}{z} & z \neq 0 \\
 f'(0) & z = 0
\end{cases}
\]
is holomorphic in \( \mathbb{D} \), so if \( |g(z)| \) is constant, then \( g(z) \) is constant as well. (This follows from the Cauchy-Riemann equations.) Hence, \( \frac{f(z)}{z} = e^{i\theta} \) for some real \( \theta \), and \( f(z) = e^{i\theta} z \) is a rotation of the disk. \( \square \)

**Exercise 6.** Use Rouché’s theorem to prove the fundamental theorem of algebra.

**Proof.** It suffices to prove the theorem for monic polynomials, as dividing by a leading term does not change roots. So let \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) be a polynomial and set \( g(z) = z^n \). Then on the boundary of a disk with sufficiently large radius \( R \),
\[
|f(z) - g(z)| = |a_{n-1}z^{n-1} + \cdots + a_1z + a_0| \\
\leq |a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0| \\
= |a_{n-1}|R^{n-1} + \cdots + |a_1|R + |a_0| \\
< R^n \quad \text{if } R \text{ is large} \\
= |z|^n \\
= |g(z)|.
\]

Thus, by Rouché’s theorem, \( f \) and \( g \) have the same number of roots in a large disk centered at 0. Since \( g \) clearly has a root at \( z = 0 \), the result follows. \( \square \)