Exercise 1. Let $f(z) = z^n$, $n \geq 1$. Check by calculation of the line integral that

$$\oint_{\gamma} f(z) \, dz = \frac{1}{n+1} (i^{n+1} - 1)$$

if $\gamma(t) = \cos t + i \sin t$, $t \in [0, \frac{\pi}{2}]$.

Proof. We see that

$$\oint_{\gamma} f(z) \, dz = \int_0^{\pi/2} (e^{it})^n ie^{it} \, dt$$

$$= i \int_0^{\pi/2} e^{i(n+1)t} \, dt$$

$$= \frac{1}{n+1} (e^{i(n+1)\pi/2} - 1)$$

$$= \frac{1}{n+1} (i^{n+1} - 1),$$

since $e^{i\pi/2} = i$. \qed

Exercise 2. Calculate from the definition of a line integral that

$$\oint_{\gamma} \frac{1}{z} \, dz = 2\pi i$$

if $\gamma(t) = \cos t + i \sin t$, $t \in [0, 2\pi]$.

Proof. We see that

$$\oint_{\gamma} \frac{1}{z} \, dz = \int_0^{2\pi} e^{-it} ie^{it} \, dt$$

$$= i \int_0^{2\pi} dt$$

$$= 2\pi i,$$

as desired. \qed

Exercise 3. Calculate that if $n$ is an integer $\neq -1$, then

$$\oint_{\gamma} z^n \, dz = 0$$

if $\gamma(t) = \cos t + i \sin t$, $t \in [0, 2\pi]$. 1
Proof. If $n \neq -1$, then $f(z) = z^n$ has a primitive on $\mathbb{C} \setminus \{0\}$, namely, $F(z) = \frac{1}{n+1}z^{n+1}$. Hence,

\[
\oint_{\gamma} z^n \, dz = \oint_{\gamma} F'(z) \, dz \\
= \int_{0}^{2\pi} F'(\gamma(t))\gamma'(t) \, dt \\
= \int_{0}^{2\pi} \frac{d}{dt} F(\gamma(t)) \, dt \\
= F(\gamma(2\pi)) - F(\gamma(0)) \\
= 0
\]

since $\gamma(0) = \gamma(2\pi) = 1$. \hfill \Box

**Exercise 4.** Use Exercise 2 to show that there is no holomorphic function $F$ on $\mathbb{C} \setminus \{0\}$ such that $F'(z) = \frac{1}{z}$ for all $z \in \mathbb{C} \setminus \{0\}$.

Proof. If there were such a function $F$, then, with $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$, we would have

\[
\oint_{\gamma} \frac{1}{z} \, dz = \oint_{\gamma} F'(z) \, dz \\
= \int_{0}^{2\pi} F'(\gamma(t))\gamma'(t) \, dt \\
= \int_{0}^{2\pi} \frac{d}{dt} F(\gamma(t)) \, dt \\
= F(\gamma(2\pi)) - F(\gamma(0)) \\
= 0.
\]

However, in Exercise 2 we showed that this integral equals $2\pi i$, so there cannot exist such an $F$. \hfill \Box

**Exercise 5.** Show that the line integral of $\frac{1}{z}$ on the path going from 1 to $x$ and then from $x$ to $x + iy$ (as shown in the picture on the homework sheet) is

\[
\frac{1}{2} \log(x^2 + y^2) + i \arctan\left(\frac{y}{x}\right)
\]

by calculating the integrals directly.

Proof. Let $\gamma_1(t) = 1 + t(x-1)$, $t \in [0, 1]$, and let $\gamma_2(t) = x + tiy$, $t \in [0, 1]$. Then, if $\log(re^{i\theta}) = r + i\theta$ is the principal branch of the logarithm on $\mathbb{C} \setminus \{z \leq 0\}$, the integral
we want is
\[ \int_{\gamma_1} \frac{1}{z} \, dz + \int_{\gamma_2} \frac{1}{z} \, dz = \int_0^1 \frac{x - 1}{1 + t(x - 1)} \, dt + \int_0^1 \frac{i y}{x + t i y} \, dt \]
\[ = [\log x - \log 1] + [\log(x + i y) - \log x] \]
\[ = \log(x + i y) \]
\[ = \log \sqrt{x^2 + y^2} + i \arctan \left( \frac{y}{x} \right) \]
\[ = \frac{1}{2} \log(x^2 + y^2) + i \arctan \left( \frac{y}{x} \right), \]
as desired. \hfill \Box

Exercise 6. How is Exercise 5 related to the fact that \( f(re^{i\theta}) = \log r + i\theta \) is holomorphic with \( f'(z) = \frac{1}{z} \) on \( \{ re^{i\theta} \mid r > 0, \frac{-\pi}{2} < \theta < \frac{\pi}{2} \} \)?

Proof. The previous exercise shows that
\[ f(z) = \int_{\gamma_z} \frac{1}{z} \, dz, \]
where \( \gamma_z \) is the path in that problem with endpoint \( z \). Since \( U = \{ re^{i\theta} \mid r > 0, \frac{-\pi}{2} < \theta < \frac{\pi}{2} \} \) is simply connected, \( \frac{1}{z} \) has a primitive \( F \) on \( U \), which we obtain by integrating \( \frac{1}{z} \) along \( \gamma_z \) (see lecture notes). Thus, \( F \) is holomorphic on \( U \), and our computation from Exercise 5 shows that \( F = f \) on \( U \). \hfill \Box

Exercise 7. Show that if \( f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \) (with the series converging for all \( z \) with \( |z| = R, R > 1 \)), then for \( n \geq 0 \),
\[ a_n = \frac{1}{2\pi i} \oint_{\gamma} z^{-(n+1)} f(z) \, dz, \]
where \( \gamma(t) = e^{it}, t \in [0, 2\pi] \).

Proof. We see that
\[ \oint_{\gamma} z^{-(n+1)} f(z) \, dz = \oint_{\gamma} z^{-(n+1)} \left( \sum_{k=0}^{\infty} a_k z^k \right) \, dz \]
\[ = \sum_{k=0}^{\infty} a_k \oint_{\gamma} z^{k-n-1} \, dz \]
\[ = 2\pi i a_n, \]
by Exercises 2 and 3, since the power series for \( f \) converges uniformly on \( \{|z| = 1\} \). \hfill \Box
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