Exercise 1. Do the algebra to prove $|a+bi| + |c+di| \geq |(a+c) + (b+d)i|$.

Proof. First, we see that

$$|(a+c) + (b+d)i|^2 = (a+c)^2 + (b+d)^2$$

$$= (a^2 + b^2 + c^2 + d^2) + 2(ac + bd)$$

and

$$|(a+bi) + (c+di)|^2 = (\sqrt{a^2 + b^2 + c^2 + d^2})^2$$

$$= (a^2 + b^2 + c^2 + d^2) + 2\sqrt{(a^2 + b^2)(c^2 + d^2)}.$$

Thus, to finish the proof, we need only show that

$$\sqrt{(a^2 + b^2)(c^2 + d^2)} \geq ac + bd.$$

Note that $0 \leq (ad - bc)^2 = a^2d^2 + b^2c^2 - 2abcd$, so

$$2abcd \leq a^2d^2 + b^2c^2.$$

This implies

$$(ac + bd)^2 = a^2c^2 + b^2d^2 + 2abcd$$

$$\leq a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$

$$= (a^2 + b^2)(c^2 + d^2),$$

so

$$ac + bd \leq |ac + bd| \leq \sqrt{(a^2 + b^2)(c^2 + d^2)},$$

and we are done. \(\square\)

Exercise 2. Show that $\left|\frac{z - a}{1 - \overline{a}z}\right| = 1$ if $|a| < 1$ and $|z| = 1$ (a, z complex).
Proof. Since $|z| = 1$, we see that
\[
\left| \frac{z - a}{1 - \bar{a}z} \right|^2 = \left( \frac{z - a}{1 - \bar{a}z} \right) \left( \frac{\bar{z} - \bar{a}}{1 - a\bar{z}} \right)
= \frac{(z - a)(\bar{z} - \bar{a})}{(1 - \bar{a}z)(1 - a\bar{z})}
= \frac{|z|^2 - a\bar{z} - \bar{a}z + |a|^2}{1 - a\bar{z} - \bar{a}z + |a|^2|z|^2}
= \frac{1 - a\bar{z} - \bar{a}z + |a|^2}{1 - a\bar{z} - \bar{a}z + |a|^2} \quad (|z| = 1)
= 1,
\]
as desired. \hfill \Box

Exercise 3. Suppose $z^5 = 1$ but $z \neq 1$.

(a) Show that $z^4 + z^3 + z^2 + z + 1 = 0$.

(b) Let $w = z + \frac{1}{z}$. Show that $w^2 + w - 1 = 0$.

(c) Solve $w^2 + w - 1 = 0$ using the quadratic formula.

(d) Use the two $w$-values to find the four $z$-values that satisfy $z^4 + z^3 + z^2 + z + 1 = 0$.

(e) What are $\cos 72^\circ$, $\sin 72^\circ$?

Proof.

(a) We can factor
\[
0 = 1 - z^5 = (1 - z)(z^4 + z^3 + z^2 + z + 1).
\]
Since $z \neq 1$, we must have $z^4 + z^3 + z^2 + z + 1 = 0$.

(b) We see that
\[
w^2 + w - 1 = \left( z + \frac{1}{z} \right)^2 + z + \frac{1}{z} - 1
= z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2}
= \frac{z^4 + z^3 + z^2 + z + 1}{z^2}
= 0
\]
since $z^4 + z^3 + z^2 + z + 1 = 0$ by part (a).

(c) By the quadratic formula,
\[
w = \frac{-1 \pm \sqrt{5}}{2}.
\]
(d) Now treating $w$ as a constant and $z$ as a variable, we see that $w = z + \frac{1}{z}$ implies

$$z^2 - wz + 1 = 0.$$ 

By the quadratic formula, we find that

$$z = \frac{w \pm \sqrt{w^2 - 4}}{2}.$$ 

For $w = -1 + \sqrt{5} \over 2$, we have $w^2 = 3 - \sqrt{5} \over 2$, so

$$z = \frac{-1 + \sqrt{5}}{4} \pm \frac{1}{2} \sqrt{-5 - \sqrt{5} \over 2} = \frac{-1 + \sqrt{5}}{4} \pm i \frac{\sqrt{5 + \sqrt{5}}}{2}.$$ 

For $w = -1 - \sqrt{5} \over 2$, we have $w^2 = 3 + \sqrt{5} \over 2$, so

$$z = \frac{-1 - \sqrt{5}}{4} \pm \frac{1}{2} \sqrt{5 - 5 \over 2} = \frac{-1 - \sqrt{5}}{4} \pm i \frac{\sqrt{5 - \sqrt{5}}}{2}.$$ 

(e) Note that $72^\circ$ is $\pi / 5$ radians. If $z = e^{2\pi i / 5}$, then $z^5 = 1$ and $z \neq 1$, so $z$ is one of the four complex numbers computed in part (d). We see geometrically that $z$ must lie in the first quadrant, so we must have

$$z = \frac{-1 + \sqrt{5}}{4} + i \frac{\sqrt{5 + \sqrt{5}}}{2}.$$ 

Since $e^{2\pi i / 5} = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \cos 72^\circ + i \sin 72^\circ$, it follows that

$$\cos 72^\circ = \frac{-1 + \sqrt{5}}{4} \quad \text{and} \quad \sin 72^\circ = \frac{1}{2} \sqrt{5 + \sqrt{5} \over 2}.$$ 

\□

Exercise 4. Suppose $\Im z > 0$.

(a) Show that $\left| \frac{z - i}{z + i} \right| < 1$.

(b) Solve $w = \frac{z - i}{z + i}$ for $z$, assuming $|w| < 1$.

(c) For the $z$ from part (b), show that $\Im z > 0$.

Proof.
(a) We see that
\[
\frac{|z - i|}{|z + i|}^2 = \frac{|z|^2 + iz - i\bar{z} + 1}{|z|^2 - iz + i\bar{z} + 1}
\]
\[
= \frac{|z|^2 - 2\text{Im} z + 1}{|z|^2 + 2\text{Im} z + 1}
\]
\[
= \frac{(\text{Re} z)^2 + (\text{Im} z - 1)^2}{(\text{Re} z)^2 + (\text{Im} z + 1)^2}
\]
\[
< 1,
\]
since \(\text{Im} z > 0\).

(b) If \(w = \frac{z - i}{z + i}\), then
\[
wz + iw = z - i \implies z(w - 1) = -i(w + 1)
\]
\[
\implies z = -i \left( \frac{w + 1}{w - 1} \right).
\]

(c) We see that
\[
\text{Im} z = \frac{1}{2i}(z - \bar{z})
\]
\[
= \frac{1}{2i} \left[ -i \left( \frac{w + 1}{w - 1} \right) - i \left( \frac{\bar{w} + 1}{\bar{w} - 1} \right) \right]
\]
\[
= -\frac{1}{2} \left( \frac{2|w|^2 - 2}{|w - 1|^2} \right)
\]
\[
= \frac{1 - |w|^2}{|w - 1|^2}
\]
\[
> 0,
\]
since \(|w| < 1\). \[\Box\]

**Exercise 5.** Check that if \(z_1 \neq 0, z_2 \neq 0\), and \(z_1^2 + z_2^2 = z_1 z_2\), then \(|z_1| = |z_2|\) and the angle between \(z_1\) and \(z_2\) is 60°.

**Proof.** Since \(z_1^2 + z_2^2 = z_1 z_2\), we see that
\[
1 + \left( \frac{z_2}{z_1} \right)^2 = \frac{z_2}{z_1}.
\]
If we set \(w = \frac{z_2}{z_1}\), the resulting equation is \(1 + w^2 = w\), and the quadratic formula tells us that
\[
w = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\pi/3}.
\]
Hence $z_2 = e^{\pm \pi i/3} z_1$. It follows that $|z_2| = |z_1|$ and that the angle between $z_1$ and $z_2$ is $\frac{\pi}{3}$ radians, or 60°.

**Exercise 6.** Show that if $z_1, z_2, z_3$ are three distinct complex numbers and $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$, then $z_1, z_2,$ and $z_3$ are the vertices of an equilateral triangle.

**Proof.** We must show that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$. First, note that
\[
(z_2 - z_1)^2 + (z_3 - z_1)^2 = 2z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3
\]
\[
= z_1^2 + (z_1z_2 + z_1z_3 + z_2z_3) - 2z_1z_2 - 2z_1z_3
\]
\[
= z_1^2 - z_1z_2 - z_1z_3 + z_2z_3
\]
\[
= (z_2 - z_1)(z_3 - z_1).
\]
By Exercise 5, it follows that $|z_2 - z_1| = |z_3 - z_1|$. Similarly, one can show that
\[
(z_1 - z_2)^2 + (z_3 - z_2)^2 = (z_1 - z_2)(z_3 - z_2),
\]
and by Exercise 5 we have $|z_2 - z_1| = |z_3 - z_2|$. Thus, the line segments between $z_1$, $z_2$, and $z_3$ all have the same length, so they form an equilateral triangle. □