Selected Solutions from Thomas S. Ferguson's *Game Theory*

Jeffrey Lee Hellrung, Jr.

April 03, 2009

1.1.3 (a) The strategy of the previous exercise will not work. You would begin by choosing a 3, and each time your opponent chooses a 4, your response is to choose another 3. We thus get the sequence of moves (I)3, (II)4, (I)3, (II)4, (I)3, (II)4, (I)3, (II)4, bringing the total to 28 with your move next. The previous exercise's strategy now is, again, to choose a 3, bringing the total to exactly 31, but this is not possible since four 3's have already been used. It is easy to see that, at this point, your opponent will win.

(b) The key observation is that $7n + 3$ is a P-position if there are at least $4 - n$ of each card left. Thus, 3, 10, 17, and 24 are P-positions if, respectively, at least 4, 3, 2, and 1 of each card remain. Using this observation, we discover that an optimal first move is to choose a 5:

```
     I         II             I          II             I          II             I          II             I
            +--A--  6 --4-- 10*
            |--2--  7 --3-- 10*
0 --5-- 5 --+--3--  8 --2-- 10*  +--A-- 13 --4-- 17*
            |--4--  9 --A-- 10*  |--2-- 14 --3-- 17*
            |--5-- 10 --2-- 12 --+--3-- 15 --2-- 17*  +--A-- 20 --4-- 24*
            +--6-- 11 --6-- 17*  |--4-- 16 --A-- 17*  |--2-- 21 --3-- 24*
                                 |--5-- 17 --2-- 19 --+--3-- 22 --2-- 24*  +--A-- 27 --4-- 31**
                                 +--6-- 18 --6-- 24*  |--4-- 23 --A-- 24*  |--2-- 28 --3-- 31**
                                                      |--5-- 24 --2-- 26 --+--3-- 29 --2-- 31**
                                                      +--6-- 25 --6-- 31** |--4-- 30 --A-- 31**
                                                                           +--6-- 32**
```

Those positions marked with a single asterisk (*) are P-positions by the above observation; those marked with double asterisk (**) are trivially P-positions or N-positions. Thus, no matter what player II plays, player I wins.

It was easy to see that choosing a 4 or 6 is *not* an optimal first move for player I, and will result in a win for player II. In both cases, player II may bring the total to 10 on his or her turn with at least 3 of each card left, which is a P-position.

Choosing a 3 is also not an optimal first move for player I. The analysis is similar to the case of choosing a 5 initially, except we start from player II's optimial response, which is to choose a 4:

```
     I         II        I               II        I               II        I               II             I          II
                    +--A--  8 --2-- 10*  +--A-- 15 --2-- 17*  +--A-- 22 --2-- 24*  +--A-- 29 --2-- 31**
                    |--2--  9 --A-- 10*  |--2-- 16 --A-- 17*  |--2-- 23 --A-- 24*  |--2-- 30 --A-- 31**
0 --3-- 3 --4-- 7 --+--3-- 10 --4-- 14 --+--3-- 17 --4-- 21 --+--3-- 24 --4-- 28 --+--4-- 32**
                    |--4-- 11 --6-- 17*  |--4-- 18 --6-- 24*  |--4-- 25 --6-- 31** |--5-- 33**
                    |--5-- 12 --5-- 17*  |--5-- 19 --5-- 24*  |--5-- 26 --5-- 31** +--6-- 34**
                    +--6-- 13 --4-- 17*  +--6-- 20 --4-- 24*  +--6-- 27 --4-- 31**
```

The analysis for choosing an A or 2 initially is more complex. If someone can show that these are optimal or non-optimal, let me know.

1.1.6 (a) We work backward, beginning with the terminal position, which is a P-position:

`[X]`

The following positions can be reached in one move to the terminal position, hence are N-positions:

```
         [ ]
[ ]      [ ]
[X]      [X]      [X][ ]
```

From these, we get the following P-position. Any move from this results in one of the above N-positions:

```
[ ]
[X][ ]
```

The following positions can be reached in one move to the above P-position, hence are N-positions:

```
[ ]
[ ]         [ ][ ]
[X][ ]      [X][ ]
```

Finally, we get the following P-position:

```
[ ]
[ ][ ]
[X][ ]
```

Since the given position in the problem can be moved to the above P-position (by chomping at $(3, 1)$), the given position is an N-position.

(b) Suppose player I initially chomps the upper right corner. If the resulting position is a P-position, then it follows that the initial position was an N-position. On the other hand, if the resulting position is an N-position, then there exists a move to a P-position for player II. This move, however, is also available to player I initially, hence the initial position is, again, an N-position. In either case, player I wins with optimal play.

1.1.7 (a) One can describe the state of the game with an ordered pair of natural numbers (n, k), where n represents the number of chips remaining and k represents the maximum number of chips that may be taken on the next move. From this, we may construct a table of N-positions and P-positions:

$k \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	P	N	P	N	P	N	P	N	P	N	P	N	P
2	P	N	N	N	P	N	N	N	P	N	N	N	P
3	P	N	N	N	P	N	N	N	P	N	N	N	P
4	P	N	N	N	N	N	N	N	P	N	N	N	N
5	P	N	N	N	N	N	N	N	P	N	N	N	N
6	P	N	N	N	N	N	N	N	P	N	N	N	N
7	P	N	N	N	N	N	N	N	P	N	N	N	N
8	P	N	N	N	N	N	N	N	N	N	N	N	N

From the above empirical evidence, we conjecture that (n, k) is a P-position if and only if $k < 2^q$ for some $2^q | n$; in particular, if $n > 0$ we may take 2^q to be the highest power-of-2 dividing n. To prove this claim, note that this is consistent with the terminal positions $n = 0$ and for any (n, k) where $n \leq k$ (i.e., the next player is allowed to take the whole pile). Now assume the claim for all followers of (n, k), where $0 < k < n$. Let 2^q be the highest power-of-2 dividing n, so $n = 2^q(2m + 1)$ for some m. We consider two cases:

* If $k \geq 2^q$, then removing 2^q chips from the pile of n chips is a valid move. The size of the remainder, $n - 2^q = 2^{q+1}m$, is divisible by 2^{q+1}, which is greater than 2^q. It follows, by the induction hypothesis, that $(n - 2^q, 2^q)$ is a P-position, hence (n, k) is an N-position and an optimal move is to take 2^q chips from the pile.

* If $k < 2^q$, then consider one of the followers $(n-j, j)$ of (n, k), $1 \le j \le k$. The highest power-of-2 that can divide $n-j$ must be less than j, since $j \le k < 2^q$ and 2^q divides n. Thus, $(n-j, j)$ is an N-position by the induction hypothesis, for any valid j, hence (n, k) is a P-position.

We conclude that (n, k) is a P-position if and only if $k < 2^q$, as desired.

It follows that if $(n, k) = (44, 43)$, the optimal move for player I is to take 4 chips.

1.2.4. (a) A move consists of turning a H to a T and, optionally, turning over one other coin to the left. Thus, we may identify:

* the turning of only coin n from H to T as removing the pile of n chips;
* the turning of coin n from H to T and the turning of coin k from T to H as removing $n-k$ chips from the pile of n chips, leaving k chips; and
* the turning of coin n from H to T and the turning of coin k from H to T as removing $n-k$ chips from the pile of n chips, leaving 2 piles of k chips each.

The last identification may be made since 2 piles of equal size in nim effectively cancel each other out, as long as each player plays optimally. In other words, a play on one of the two piles can be exactly mirrored by the next player with a play on the other pile.

(b) The H's are at positions 2, 5, 9, 10, and 12, giving a nim sum of

$$2 \oplus 5 \oplus 9 \oplus 10 \oplus 12 = ((2 \oplus 5) \oplus (9 \oplus 10)) \oplus 12 = (7 \oplus 3) \oplus 12 = 4 \oplus 12 = 8.$$

A winning move could then be any of the following:

* Flip coin 9 to T and coin $9 \oplus 8 = 1$ to H.
* Flip coin 10 to T and coin $10 \oplus 8 = 2$ to T.
* Flip coin 12 to T and coin $12 \oplus 8 = 4$ to H.

1.2.6. The claim is that $(x_1, x_2, \ldots, x_n)$ is a P-position in staircase nim (SCN P-position) if and only if $(x_1, x_3, \ldots, x_k)$ is a P-position in nim (nim P-position), where $k = n$ if n is odd and $k = n-1$ if n is even. The claim is certainly true for the terminal position of staircase nim. So let $(x_1, x_2, \ldots, x_n)$ be a non-terminal position in staircase nim, and suppose all followers of $(x_1, x_2, \ldots, x_n)$ are N-positions or P-positions according to the claim. We consider two cases:

- If $(x_1, x_3, \ldots, x_k)$ is a nim N-position, then there exists some move in nim to a nim P-position, say, removing x coins from the pile corresponding to x_j (j being odd), giving a nim P-position of $(x_1, x_3, \ldots, x_j - x, \ldots, x_k)$.
  * If $j = 1$, then, by the induction hypothesis, $(x_1 - x, x_2, \ldots, x_n)$ is a SCN P-position, and this position can be reached from $(x_1, x_2, \ldots, x_n)$ by removing x coins from step 1.
  * If $j > 1$, then, by the induction hypothesis, $(x_1, x_2, \ldots, x_{j-1} + x, x_j - x, \ldots, x_n)$ is a SCN P-position, and this position can be reached from $(x_1, x_2, \ldots, x_n)$ by moving x coins from step j to step $j-1$.

  In both cases, one can move to a SCN P-position, hence $(x_1, x_2, \ldots, x_n)$ is a SCN N-position.
- If $(x_1, x_3, \ldots, x_k)$ is a nim P-position, then every follower of $(x_1, x_3, \ldots, x_k)$ in nim is a nim N-position. By the induction hypothesis, then, every follower of $(x_1, x_2, \ldots, x_n)$ in staircase nim is a SCN N-position, hence $(x_1, x_2, \ldots, x_n)$ is a SCN P-position.

We conclude that $(x_1, x_2, \ldots, x_n)$ is a SCN P-position if and only if $x_1, x_3, \ldots, x_k$ is a nim P-position.

1.3.3 We may construct a table of Sprague-Grundy values $g(x)$ based on the size x of the pile:

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(x)$	0	0	1	0	2	1	3	0	4	2	5	1	6	3	7	0

From the above empirical evidence, we conjecture that

$$g(x) = \begin{cases} x/2, & x \text{ even} \\ g((x-1)/2), & x \text{ odd} \end{cases}.$$

To prove this, let $h(x)$ denote the right-hand side, and observe that $g(x) = 0 = h(x)$ for the terminal positions $x = 0$ and $x = 1$. So let x be a non-terminal position, and suppose $g(y) = h(y)$ for all $y < x$. We consider two cases:

- If x is even, we note that

$$F(x) = \{x/2, x/2+1, \ldots, x-3, x-2, x-1\},$$
$$F(x-2) = \{x/2-1, x/2, x/2+1, \ldots, x-3\}.$$

  That is, the followers $F(x)$ of x are the same as the followers $F(x-2)$ of $x-2$ except for the exclusion of $x/2-1$ and the inclusion $x-2$ and $x-1$. Now, by the induction hypothesis, $g(x-2) = (x-2)/2 = x/2-1$, which implies (using the minimal excludant definition of g) that

$$g(\{x/2-1, \ldots, x-3\}) = g(F(x-2)) \supset \{0, 1, \ldots, x/2-2\}.$$

  Indeed, since the size of $F(x-2)$ is precisely $x/2-1$, we must have equality, and g restricted to $F(x-2)$ is a bijection onto $\{0, 1, \ldots, x/2-2\}$. Thus,

$$g(\{x/2, \ldots, x-3\}) = \{0, 1, \ldots, x/2-2\} \backslash \{g(x/2-1)\},$$

  and

$$g(\{x/2, \ldots, x-3, x-1\}) = \{0, 1, \ldots, x/2-2\}$$

  since, by the induction hypothesis, $g(x-1) = g(x/2-1)$. Recalling that $g(x-2) = x/2-1$, we finally obtain

$$g(F(x)) = g(\{x/2, \ldots, x-3, x-2, x-1\}) = \{0, 1, \ldots, x/2-2, x/2-1\},$$

  and hence

$$g(x) = \operatorname{mex} g(F(x)) = x/2 = h(x).$$

- If x is odd, we note that

$$F(x) = \{(x+1)/2, (x+3)/2, \ldots, x-2, x-1\},$$
$$F(x-1) = \{(x-1)/2, (x+1)/2, (x+3)/2, \ldots, x-2\}.$$

  Using a similar analysis as in the previous case, we have that $g(x-1) = (x-1)/2$,

$$g(F(x-1)) = \{0, 1, \ldots, (x-3)/2\},$$

  and, indeed, g restricted to $F(x-1)$ is a bijection onto $\{0, 1, \ldots, (x-3)/2\}$. Adding $x-1$ and removing $(x-1)/2$ from $F(x-1)$ "transforms" $F(x-1)$ into $F(x)$, giving

$$g(F(x)) = \{0, 1, \ldots, (x-1)/2\} \backslash \{g((x-1)/2)\}.$$

  Thus,

$$g(x) = \operatorname{mex} g(F(x)) = g((x-1)/2) = h(x).$$

We conclude that $g(x) = h(x)$, as desired.

1.4.5. We may construct a table of Sprague-Grundy values $g(x)$ based on the size x of a single pile:

x	$F(x)$	$g(F(x))$	$g(x)$
0	$\emptyset$	$\emptyset$	0
1	0	0	1
2	$0, 1$	$0, 1$	2
3	$1, 2, (1, 1)$	$1, 2, 0$	3
4	$2, 3, (1, 2)$	$2, 3, 3$	0
5	$3, 4, (1, 3), (2, 2)$	$3, 0, 2, 0$	1
6	$4, 5, (1, 4), (2, 3)$	$0, 1, 1, 1$	2
7	$5, 6, (1, 5), (2, 4), (3, 3)$	$1, 2, 0, 2, 0$	3
8	$6, 7, (1, 6), (2, 5), (3, 4)$	$2, 3, 3, 3, 3$	0
9	$7, 8, (1, 7), (2, 6), (3, 5), (4, 4)$	$3, 0, 2, 0, 2, 0$	1
10	$8, 9, (1, 8), (2, 7), (3, 6), (4, 5)$	$0, 1, 1, 1, 1, 1$	2
11	$9, 10, (1, 9), (2, 8), (3, 7), (4, 6), (5, 5)$	$1, 2, 0, 2, 0, 2, 0$	3
12	$10, 11, (1, 10), (2, 9), (3, 8), (4, 7), (5, 6)$	$2, 3, 3, 3, 3, 3, 3$	0

From the above empirical evidence, we conjecture that $g(x) = x \bmod 4$. This is clearly the case for the terminal position $x = 0$, so let x be some non-terminal position, and suppose that $g(y) = y \bmod 4$ for all $y < x$. We consider four cases:

- $x \equiv_4 0$: By the induction hypothesis, $g(x-1) = 3$ and $g(x-2) = 2$. Also, if $y + z = x - 1 \equiv_4 3$, then, by the induction hypothesis,

  $$g(y) + g(z) = (y \bmod 4) + (z \bmod 4) \equiv_4 y + z \equiv_4 3.$$

  Thus, $g(y) \neq g(z)$, so that $g(y) \oplus g(z) \neq 0$. This means that 0 is not among the Sprague-Grundy values of the followers $F(x)$ of x, so $g(x) = 0$.
- $x \equiv_4 1$: By the induction hypothesis, $g(x-1) = 0$ and $g(x-2) = 3$. As above, if $y+z = x-1 \equiv_4 0$, then $g(y)+g(z) \equiv_4 0$. Of course, this can only happen if $(g(y), g(z))$ is one of $(0, 0)$, $(1, 3)$, $(2, 2)$, or $(3, 1)$, all of which satisfy $g(y) \oplus g(z) \in \{0, 2\}$. This means that 1 is not among the Sprague-Grundy values of the followers $F(x)$ of x, and 0 is, so $g(x) = 1$.
- $x \equiv_4 2$: By the induction hypothesis, $g(x-1) = 1$ and $g(x-2) = 0$. As above, if $y+z = x-1 \equiv_4 1$, then $g(y) + g(z) \equiv_4 1$. Again, this can only happen if $(g(y), g(z))$ is one of $(0, 1)$, $(1, 0)$, $(2, 3)$, or $(3, 2)$, all of which satisfy $g(y) \oplus g(z) = 1$. This means that the Sprague-Grundy values of the followers $F(x)$ of x consist precisely of $\{0, 1\}$, so $g(x) = 2$.
- $x \equiv_4 3$: By the induction hypothesis, $g(x-1) = 2$ and $g(x-2) = 1$. As above, if $y+z = x-1 \equiv_4 2$, then $g(y) + g(z) \equiv_4 2$. Again, this can only happen if $(g(y), g(z))$ is one of $(0, 2)$, $(1, 1)$, $(2, 0)$, or $(3, 3)$, all of which satisfy $g(y) \oplus g(z) \in \{0, 2\}$. This means that the Sprague-Grundy values of the followers $F(x)$ of x consist precisely of $\{0, 1, 2\}$, so $g(x) = 3$.

We conclude that $g(x) = x \bmod 4$, as desired.

1.4.6 We may construct a table of Sprague-Grundy values $g(x)$ based on the size x of a single pile:

x	$F(x)$	$g(F(x))$	$g(x)$
0	$\emptyset$	$\emptyset$	0
1	0	0	1
2	0	0	1
3	$0, 1$	$0, 1$	2
4	$F(3), 2, (1, 1)$	$\{0, 1\}, 1, 0$	2
5	$F(4), 3, (1, 2)$	$\{0, 1\}, 2, 0$	3
6	$F(5), 4, (1, 3), (2, 2)$	$\{0, 1, 2\}, 2, 3, 0$	4
7	$F(6), 5, (1, 4), (2, 3)$	$\{0, 1, 2, 3\}, 3, 3, 3$	4
8	$F(7), 6, (1, 5), (2, 4), (3, 3)$	$\{0, 1, 2, 3\}, 4, 2, 3, 0$	5
9	$F(8), 7, (1, 6), (2, 5), (3, 4)$	$\{0, \ldots, 4\}, 4, 5, 2, 0$	6
10	$F(9), 8, (1, 7), (2, 6), (3, 5), (4, 4)$	$\{0, \ldots, 5\}, 5, 5, 5, 1, 0$	6
11	$F(10), 9, (1, 8), (2, 7), (3, 6), (4, 5)$	$\{0, \ldots, 5\}, 5, 4, 5, 6, 1$	7
12	$F(11), 10, (1, 9), \ldots, (5, 5)$	$\{0, \ldots, 6\}, 6, 7, 4, 6, 6, 0$	8
13	$F(12), 11, (1, 10), \ldots, (5, 6)$	$\{0, \ldots, 7\}, 7, 7, 7, 7, 6, 7$	8

From the above empirical evidence, we conjecture that

$$g(x) = \begin{cases} 2x/3, & x \equiv_3 0 \\ 2(x-1)/3, & x \equiv_3 1, x \neq 1 \\ (2x-1)/3, & x \equiv_3 2 \\ 1, & x = 1 \end{cases}.$$

To prove this, let $h(x)$ denote the right-hand side. Observe that $g(x) = h(x)$ for $0 \leq x \leq 3$, so let $x > 3$, and suppose $g(y) = h(y)$ for all $y < x$. We will actually suppose (and likewise prove) something stronger. Specifically, we assume that $g(F(y)) = \{0, \ldots, h(y) - 1\}$ for all $y < x$. We begin by noting that

$$F(x) = F(x-1) \cup \{x-2\} \cup \bigcup_{i=1}^{x-3} \{(i, x-i-2)\}$$

and hence, by the induction hypothesis,

$$g(F(x)) = \{0, \ldots, g(x-1) - 1\} \cup \{g(x-2)\} \cup \bigcup_{i=1}^{x-3} \{g(i) \oplus g(x-i-2)\}.$$

We consider three cases:

- $x \equiv_3 0$: We have $x - 1 \equiv_3 2$, $x - 2 \equiv_3 1$, and $x - 3 \equiv_3 0$, hence

  $$g(x-1) = 2x/3 - 1, \quad g(x-2) = 2x/3 - 2, \quad g(x-3) = 2x/3 - 2,$$

  from which it follows that

  $$g(F(x-1) \cup \{x-2\}) = \{0, \ldots, 2x/3 - 2\}.$$

  Now, if we take 2 chips from a pile of size x and split it into two piles of size $x - 3$ and 1, we get a Sprague-Grundy value of $g(x-3) \oplus g(1) = (2x/3 - 2) \oplus 1 = 2x/3 - 1$ (since $2x/3 - 2$ is even). We will show that any other splitting, say into piles of size $x - i - 2$ and i for $1 < i < x - 3$, gives a Sprague-Grundy value at most $2x/3 - 1$. This will follow from the fact that $g(x-i-2) \oplus g(i) \leq g(x-i-2) + g(i)$. We consider three cases:

  * $i \equiv_3 0$: $x - i - 2 \equiv_3 1$, so $g(x-i-2) + g(i) = 2(x-i-3)/3 + 2i/3 = 2x/3 - 1$.
  * $i \equiv_3 1$: $x - i - 2 \equiv_3 0$, so $g(x-i-2) + g(i) = 2(x-i-2)/3 + 2(i-1)/3 = 2x/3 - 1$.

  - $i \equiv_3 2$: $x - i - 2 \equiv_3 2$, so $g(x - i - 2) + g(i) = (2(x - i - 2) - 1)/3 + (2i - 1)/3 = 2x/3 - 2$.

  Thus, $g(F(x)) = \{0, \ldots, 2x/3 - 1\}$, and hence $g(x) = 2x/3 = h(x)$.

- $x \equiv_3 1$: We have $x - 1 \equiv_3 0$, $x - 2 \equiv_3 2$, and $x - 3 \equiv_3 1$, hence

$$g(x - 1) = 2(x - 1)/3, \quad g(x - 2) = 2(x - 1)/3 - 1, \quad g(x - 3) = 2(x - 1)/3 - 2,$$

  from which it follows that

$$g(F(x - 1) \cup \{x - 2\}) = \{0, \ldots, 2(x - 1)/3 - 1\}.$$

  As before, we show that, if we take 2 chips from a pile of size x and split it into two piles of size $x - i - 2$ and i for $1 \leq i \leq x - 3$, we get a Sprague-Grundy value of at most $2(x - 1)/3 - 1$. We consider five cases this time (to account for the exceptional cases $i = 1$ and $i = x - 3$):

  - $i = 1$: $g(x - 3) + g(1) = 2(x - 4)/3 + 1 = 2(x - 1)/3 - 1$.
  - $i = x - 3$: Same as above.
  - $i \equiv_3 0$: $x - i - 2 \equiv_3 2$, so $g(x - i - 2) + g(i) = (2(x - i - 2) - 1)/3 + 2i/3 = 2(x - 1)/3 - 1$.
  - $i \equiv_3 1$: $x - i - 2 \equiv_3 1$, so $g(x - i - 2) + g(i) = 2(x - i - 3)/3 + 2(i - 1)/3 = 2(x - 1)/3 - 2$.
  - $i \equiv_3 2$: $x - i - 2 \equiv_3 0$, so $g(x - i - 2) + g(i) = 2(x - i - 2)/3 + (2i - 1)/3 = 2(x - 1)/3 - 1$.

  Thus, $g(F(x)) = \{0, \ldots, 2(x - 1)/3 - 1\}$, and hence $g(x) = 2(x - 1)/3 = h(x)$.

- $x \equiv_3 2$: We have $x - 1 \equiv_3 1$, $x - 2 \equiv_3 0$, and $x - 3 \equiv_3 2$, hence

$$g(x - 1) = (2x - 1)/3 - 1, \quad g(x - 2) = (2x - 1)/3 - 1, \quad g(x - 3) = (2x - 1)/3 - 2,$$

  from which it follows that

$$g(F(x - 1) \cup \{x - 2\}) = \{0, \ldots, (2x - 1)/3 - 1\}.$$

  As before, we show that, if we take 2 chips from a pile of size x and split it into two piles of size $x - i - 2$ and i for $1 \leq i \leq x - 3$, we get a Sprague-Grundy value of at most $(2x - 1)/3 - 1$. We consider five cases this time (to account for the exceptional cases $i = 1$ and $i = x - 3$):

  - $i = 1$: $g(x - 3) + g(1) = (2x - 1)/3 - 2 + 1 = (2x - 1)/3 - 1$.
  - $i = x - 3$: Same as above.
  - $i \equiv_3 0$: $x - i - 2 \equiv_3 0$, so $g(x - i - 2) + g(i) = 2(x - i - 2)/3 + 2i/3 = (2x - 1)/3 - 1$.
  - $i \equiv_3 1$: $x - i - 2 \equiv_3 2$, so $g(x - i - 2) + g(i) = (2(x - i - 2) - 1)/3 + 2(i - 1)/3 = (2x - 1)/3 - 2$.
  - $i \equiv_3 2$: $x - i - 2 \equiv_3 1$, so $g(x - i - 2) + g(i) = 2(x - i - 3)/3 + (2i - 1)/3 = (2x - 1)/3 - 2$.

  Thus, $g(F(x)) = \{0, \ldots, (2x - 1)/3 - 1\}$, and hence $g(x) = (2x - 1)/3 = h(x)$.

We conclude that $g(x) = h(x)$, as desired.

1.4.7. We may construct a table of Sprague-Grundy values $g(x)$ based on the size x of a single pile:

x	$F(x)$	$g(F(x))$	$g(x)$
0	$\emptyset$	$\emptyset$	0
1	0	0	1
2	1	1	0
3	$2, (1,1)$	$0, 0$	1
4	$F(1), 3, (1,2)$	$\{0\}, 1, 1$	2
5	$F(2), 4, (1,3), (2,2)$	$\{1\}, 2, 0, 0$	3
6	$F(3), 5, (1,4), (2,3)$	$\{0,0\}, 3, 3, 1$	2
7	$F(4), 6, (1,5), (2,4), (3,3)$	$\{0,1\}, 2, 2, 2, 0$	3
8	$F(5), 7, (1,6), (2,5), (3,4)$	$\{0,1,2\}, 3, 3, 3, 3$	4
9	$F(6), 8, (1,7), (2,6), (3,5), (4,4)$	$\{0,1,3\}, 4, 2, 2, 2, 0$	5
10	$F(7), 9, (1,8), (2,7), (3,6), (4,5)$	$\{0,1,2\}, 5, 5, 3, 3, 1$	4
11	$F(8), 10, (1,9), (2,8), (3,7), (4,6), (5,5)$	$\{0,1,2,3\}, 4, 4, 4, 2, 0, 0$	5
12	$F(9), 11, (1,10), (2,9), (3,8), (4,7), (5,6)$	$\{0,\ldots,4\}, 5, 5, 5, 5, 1, 1$	6
13	$F(10), 12, (1,11), (2,10), (3,9), (4,8), (5,7), (6,6)$	$\{0,1,2,3,5\}, 6, 4, 4, 4, 6, 0, 0$	7
14	$F(11), 13, (1,12), (2,11), (3,10), (4,9), (5,8), (6,7)$	$\{0,\ldots,4\}, 7, 7, 5, 5, 7, 7, 1$	6
15	$F(12), 14, (1,13), (2,12), (3,11), (4,10), (5,9), (6,8), (7,7)$	$\{0,\ldots,5\}, 6, 6, 6, 4, 6, 6, 6, 0$	7

From the above empirical evidence, we conjecture that

$$g(x) = \begin{cases} x/2, & x \equiv_4 0 \\ (x+1)/2, & x \equiv_4 1 \\ x/2 - 1, & x \equiv_4 2 \\ (x-1)/2, & x \equiv_4 3 \end{cases}.$$

1.5.2. For the following games, let $M(n)$ denote the maximum number of moves of the game beginning with a single head at position n (with the left-most coin at position 1). We observe that it is always better (in the context of maximizing the number of moves) to turn over as many T's to H's as possible as far to the right as possible. Although these may be competing conditions in general, for the following games, both these conditions may be satisfied. Also, the maximum number of moves from a position with multiple H's is the sum of the $M(n_i)$, where $\{n_i\}$ is set of positions with H's. Thus, in the following games, we derive a recurrence relation that M must satisfy, then state the solution of the recurrence relation.

(a) $M(n) = 1 + M(n-1)$, and $M(1) = 1$, hence $M(n) = n$.

(b) $M(n) = 1 + M(n-1) + M(n-2)$, and $M(1) = 1$ and $M(2) = 2$. It may not be immediately obvious what the solution to the recurrence is, but if we rewrite it as $M(n) + 1 = (M(n-1) + 1) + (M(n-2) + 1)$, and then let $M'(n) = M(n) + 1$, we find that $M'(n) = M'(n-1) + M'(n-2)$, with $M'(1) = 2$ and $M'(2) = 3$, so $M(n) = F_{n+2}$, where F_n is the n^{th} Fibonacci number.

(c) $M(n) = 1 + \sum_{k=1}^{n-1} M(k)$, and $M(1) = 1$, hence $M(n) = 2^{n-1}$.

1.5.4. We may construct a table of Sprague-Grundy values $g(x)$, where x denotes a single H at position x, with $x = 1$ being the left-most position (as with Example 4):

x	$F(x)$	$g(F(x))$	$g(x)$
1	$\emptyset$	$\emptyset$	0
2	1	0	1
3	$2, (1,2)$	$1, 1$	0
4	$3, (2,3), (1,2,3)$	$0, 1, 1$	2
5	$4, (3,4), (2,3,4), (1,2,3,4)$	$2, 2, 3, 3$	0
6	$5, \ldots, (1, \ldots, 5)$	$0, 2, 2, 3, 3$	1
7	$6, \ldots, (1, \ldots, 6)$	$1, 1, 3, 3, 2, 2$	0
8	$7, \ldots, (1, \ldots, 7)$	$0, 1, 1, 3, 3, 2, 2$	4
9	$8, \ldots, (1, \ldots, 8)$	$4, 4, 5, 5, 7, 7, 6, 6$	0
10	$9, \ldots, (1, \ldots, 9)$	$0, 4, 4, 5, 5, 7, 7, 6, 6$	1
11	$10, \ldots, (1, \ldots, 10)$	$1, 1, 5, 5, 4, 4, 6, 6, 7, 7$	0
12	$11, \ldots, (1, \ldots, 11)$	$0, 1, 1, 5, 5, 4, 4, 6, 6, 7, 7$	2
13	$12, \ldots, (1, \ldots, 12)$	$2, 2, 3, 3, 7, 7, 6, 6, 4, 4, 5, 5$	0
14	$13, \ldots, (1, \ldots, 13)$	$0, 2, 2, 3, 3, 7, 7, 6, 6, 4, 4, 5, 5$	1
15	$14, \ldots, (1, \ldots, 14)$	$1, 1, 3, 3, 2, 2, 6, 6, 7, 7, 5, 5, 4, 4$	0
16	$15, \ldots, (1, \ldots, 15)$	$0, 1, 1, 3, 3, 2, 2, 6, 6, 7, 7, 5, 5, 4, 4$	8

From the above empirical evidence, we conjecture that

$$g(x) = \begin{cases} g_{\text{Ruler}}(x/2), & x \text{ even} \\ 0, & x \text{ odd} \end{cases}.$$

1.5.5. (a)

$$\begin{aligned} 6 \otimes 21 &= (4 \oplus 2) \otimes (16 \oplus 4 \oplus 1) \\ &= (4 \otimes 16) \oplus (4 \otimes 4) \oplus (4 \otimes 1) \oplus (2 \otimes 16) \oplus (2 \otimes 4) \oplus (2 \otimes 1) \\ &= 64 \oplus 6 \oplus 4 \oplus 32 \oplus 8 \oplus 2 \\ &= 104 \end{aligned}$$

(b)

$$\begin{aligned} 25 \otimes 40 &= (16 \oplus 8 \oplus 1) \otimes (2 \otimes 16 \oplus 8) \\ &= (2 \otimes 16 \otimes 16) \oplus (16 \otimes 8) \oplus (2 \otimes 8 \otimes 16) \oplus (8 \otimes 8) \oplus 40 \\ &= (2 \otimes 16 \otimes 16) \oplus 128 \oplus (2 \otimes 8 \otimes 16) \oplus 13 \oplus 40 \\ &= (2 \otimes 16 \otimes 16) \oplus (2 \otimes 8 \otimes 16) \oplus 165; \\ 2 \otimes (16 \otimes 16) &= 2 \otimes 24 \\ &= 2 \otimes (16 \oplus 8) \\ &= (2 \otimes 16) \oplus (2 \otimes 8) \\ &= 32 \oplus 12 \\ &= 44; \\ (2 \otimes 8) \otimes 16 &= 12 \otimes 16 \\ &= 192. \end{aligned}$$

Thus,

$$25 \otimes 40 = 44 \oplus 192 \oplus 165 = 73.$$

(c) From Table 5.2, we find that $13 \otimes 14 = 1$, hence $14^{-1} = 13$; also from Table 5.2, $15 \otimes 13 = 12$. Thus $15 \oslash 14 = 15 \otimes 13 = 12$.

(d) From Table 5.2, $14 \otimes 14 = 8$, so $\sqrt{8} = 14$.

(e) We seek a and b such that

$$(x \oplus a) \otimes (x \oplus b) = x^2 \oplus (a \oplus b) \otimes x \oplus a \otimes b = x^2 \oplus x \oplus 6,$$

i.e., $a \oplus b = 1$ and $a \otimes b = 6$. A search through Table 5.2 gives $a, b = 14, 15$.

1.5.8. (a) Let g_{MT} and g_{R} be the Sprague-Grundy functions for Mock Turtles and Ruler, respectively. Then the table of Sprague-Grundy values for (Mock Turtles) $\times$ (Ruler) is as follows (using Table 5.2 and Examples 3 and 4):

	y	1	2	3	4	5	6	7	8
x	$g_{\mathrm{MT}}(x) \backslash g_{\mathrm{R}}(y)$	1	2	1	4	1	2	1	8
0	1	1	2	1	4	1	2	1	8
1	2	2	3	2	8	2	3	2	12
2	4	4	8	4	6	4	8	4	11
3	7	7	9	7	10	7	9	7	15
4	8	8	12	8	11	8	12	8	13

(b) The given position has Sprague-Grundy value $15 = 2 \oplus 13$, so we need to take the H with Sprague-Grundy value 13 (at $(x, y) = (4, 8)$) to a Sprague-Grundy value of $13 \oplus 15 = 2$. We note that 13 is the product of the Sprague-Grundy values of 8 (from Mock Turtles) and 8 (from Ruler), i.e., $13 = 8 \otimes 8$. Following the notation in **Solving Tartan Games**, we have $v = 13$, $v_1 = 8$, $v_2 = 8$, and $u = 2$.

We first need to find a move in Turning Corners that takes a H at position $(8, 8)$, with Sprague-Grundy value 13, to a Sprague-Grundy value of 2. Such a move can be identified with the position (u_1, u_2) of the northwest flipped coin. Thus, we seek u_1, u_2 such that

$$2 = (u_1 \otimes u_2) \oplus (u_1 \otimes 8) \oplus (u_2 \otimes 8).$$

A search through all 64 possible moves in Turning Corners from a single H at $(8, 8)$ yields the possible moves $(u_1, u_2) \in \{(3, 3), (4, 6), (6, 4)\}$. Alternatively, we may rewrite our condition on u_1, u_2 as

$$2 = (u_1 \oplus 8) \otimes (u_2 \oplus 8) \oplus 13 \quad \Rightarrow \quad (u_1 \oplus 8) \otimes (u_2 \oplus 8) = 15.$$

Thus, we require the product of $u_1 \oplus 8$ and $u_2 \oplus 8$ to be 15, and since we need $u_1 < 8$ and $u_2 < 8$, we must have $u_1 \oplus 8 > 8$ and $u_2 \oplus 8 > 8$. A quick search through Table 5.2 yields 3 different factorizations of 15 with both factors larger than 8: $15 = 11 \otimes 11 = 12 \otimes 14 = 14 \otimes 12$, giving $(u_1, u_2) \in \{(3, 3), (4, 6), (6, 4)\}$, as before.

We now consider each of the possible (u_1, u_2) pairs to determine what move it gives in the original Tartan game.

* $u_1, u_2 = 3, 3$: We need a move M_{MT} in Mock Turtles that takes a single H at position $x = 4$, with Sprague-Grundy value 8, to a position with Sprague-Grundy value 3. Such a move is to turn over the coins at positions 0, 1, and 4, since $g_{\mathrm{MT}}(0) \oplus g_{\mathrm{MT}}(1) = 3$. Another such move is to turn over the coins at positions 2, 3, and 4. Similarly, we need a move M_{R} in Ruler that takes a single H at position $y = 8$, with Sprague-Grundy value 8, to a position with Sprague-Grundy value 3. Such a move is to turn over the coins at positions 6, 7, and 8, since $g_{\mathrm{R}}(6) \oplus g_{\mathrm{R}}(7) = 3$. The moves $M_{\mathrm{MT}} \times M_{\mathrm{R}}$ in (Mock Turtles) $\times$ (Ruler) thus look like

$$\begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & H \end{array} \quad \to \quad \begin{array}{cccccccc} T & H & T & T & T & H & H & H \\ T & T & T & T & T & H & H & H \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & H & H & T \end{array}$$

or

$$\begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & H \end{array} \quad \to \quad \begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & H & H & H \\ T & T & T & T & T & H & H & H \\ T & T & T & T & T & H & H & T \end{array}$$

* $u_1, u_2 = 4, 6$: Similar to above, we get $M_{\text{MT}} = \{2, 4\}$ and $M_{\text{R}} = \{4, 5, 6, 7, 8\}$, giving

$$\begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & H \end{array} \quad \to \quad \begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & H & H & H & H & H \\ T & T & T & T & T & T & T & T \\ T & T & T & H & H & H & H & T \end{array}$$

* $u_1, u_2 = 6, 4$: Similar to above, we get $M_{\text{MT}} = \{0, 3, 4\}$ or $M_{\text{MT}} = \{1, 2, 4\}$ (the move for Mock Turtles is not unique) and $M_{\text{R}} = \{1, 2, 3, 4, 5, 6, 7, 8\}$. In the first Mock Turtle move, we get

$$\begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & H \end{array} \quad \to \quad \begin{array}{cccccccc} H & T & H & H & H & H & H & H \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ H & H & H & H & H & H & H & H \\ H & H & H & H & H & H & H & T \end{array}$$

The second Mock Turtle move gives

$$\begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & H \end{array} \quad \to \quad \begin{array}{cccccccc} T & H & T & T & T & T & T & T \\ H & H & H & H & H & H & H & H \\ H & H & H & H & H & H & H & H \\ T & T & T & T & T & T & T & T \\ H & H & H & H & H & H & H & T \end{array}$$

2.2.6. (a)

(b) In looking for a reduction by dominance, we notice that only column 3 can be dominated, since columns 1 and 2 contain row minimums, while all 3 rows contain column maximums. After some educated guesswork, we find that 3/8 of column 1 and 5/8 of column 2 sum to no more than column 3 (we could have also chosen 7/16 of column 1 and 9/16 of column 2, or any probability combination in between). Hence we can reduce by dominance to

$$\begin{pmatrix} 0 & 8 \\ 8 & 4 \\ 12 & -4 \end{pmatrix}.$$

If player II now chooses column 1 of this reduced game with probability q and column 2 with probabiliy $1-q$, then we can graph the expected payoff to player I for each of player I's pure strategies as functions of q. Player I's pure strategies correspond to rows 1, 2, and 3, and they have expected payoffs of, respectively, $0q+8(1-q)$; $8q+4(1-q)$; and $12q-4(1-q)$. The optimal q for player II is that which minimizes the maximum of these expected payoffs, which, by inspection of the graph, is found to be the value of q which equalizes the expected payoffs corresponding to rows 1 and 2. This further reduces the game to

$$\begin{pmatrix} 0 & 8 \\ 8 & 4 \end{pmatrix},$$

which solves to $p = q = 1/3$ and $V = 16/3$. Extrapolating back to the original 3×3 game, we have $\boldsymbol{p} = \boldsymbol{q} = (1/3, 2/3, 0)$ and $V = 16/3$.

To check our solution, we compute the expected payoff to player I for each of player I's pure strategies, given the above optimal strategy for player II:

$$A\boldsymbol{q}^T = \begin{pmatrix} 16/3 \\ 16/3 \\ 4/3 \end{pmatrix}.$$

We conclude that with strategy $\boldsymbol{q}$, player II is guaranteed an expected loss of no more than 16/3, no matter what player I does. Likewise, we compute the expected payoff to player I for each of player II's pure strategies, given the above optimal strategy for player I:

$$\boldsymbol{p}A = \begin{pmatrix} 16/3 & 16/3 & 17/3 \end{pmatrix}.$$

We conclude that with strategy $\boldsymbol{p}$, player I is guaranteed an expected payoff of at least 16/3, no matter what player II does. This confirms that the value of the game is 16/3, and $\boldsymbol{p}$ and $\boldsymbol{q}$ are optimal strategies.

2.2.6. (a) We notice that rows 3 and 5 dominate rows 1 and 7, respectively; and columns 1 and 7 dominate columns 3 and 5, respectively. Hence we can reduce by dominance to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this reduced game, row 3 and column 3 are dominated, reducing the matrix to the 4×4 identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that this reduced game has value $V = 1/4$ and optimal strategies $\boldsymbol{p} = \boldsymbol{q} = (1/4, 1/4, 1/4, 1/4)$. Extrapolating back to the original 7×7 game gives $\boldsymbol{p} = (0, 1/4, 1/4, 0, 1/4, 1/4, 0)$ and $\boldsymbol{q} = (1/4, 1/4, 0, 0, 0, 1/4, 1/4)$.

(b) First, imagine a very large $n \times n$ matrix, and focus on the upper-left corner:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Notice that, as for the 7×7 case, row 3 dominates row 1, and column 1 dominates column 3, giving the following reduced matrix:

$$\begin{pmatrix} & & & & & & \cdots \\ 1 & 0 & & 0 & 0 & 0 & \cdots \\ 0 & 1 & & 1 & 0 & 0 & \cdots \\ 0 & 0 & & 0 & 1 & 0 & \cdots \\ 0 & 0 & & 1 & 0 & 1 & \cdots \\ 0 & 0 & & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This reduced matrix can be further reduced by domination, since now row 6 dominates row 4 and column 2 dominates column 4 (numbered according to the original matrix):

$$\begin{pmatrix} & & & & & & \cdots \\ 1 & 0 & & & 0 & 0 & \cdots \\ 0 & 1 & & & 0 & 0 & \cdots \\ & & & & & & \cdots \\ 0 & 0 & & & 0 & 1 & \cdots \\ 0 & 0 & & & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, we see that the upper-left 4×4 block reduces to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and the lower-right $(n-4) \times (n-4)$ block remains unaffected, so long as $n \geq 6$. If $n \geq 10$, the lower $(n-4) \times (n-4)$ block can likewise be recursively reduced in a similar manner. Thus, when $n \geq 6$, we can reduce our original matrix such that the upper-left block is a large identity matrix, and the lower-right block is a small (5×5 or less) matrix. The resulting optimal strategies for player I and player II begin like $\boldsymbol{p} = (0, p, p, 0, 0, p, p, 0, 0, \dots)$ and $\boldsymbol{q} = (q, q, 0, 0, q, q, 0, 0, \dots)$.

It thus suffices to analyze the following 4 cases. Here, A' denotes the reduced matrix after reduction by domination in 4×4 blocks as above, and A'' denotes the final matrix after a full reduction by domination. (Note that A'' may not be unique, but the value of the game is independent of the choice of A'').

case	A'	A''
$n \equiv_4 0$	$\begin{pmatrix} I_{(n-4)/2} & & & & \\ & 0 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} I_{(n-4)/2} & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} = I_{n/2}$
$n \equiv_4 1$	$\begin{pmatrix} I_{(n-5)/2} & & & & & \\ & 0 & 1 & 0 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} I_{(n-5)/2} & & & \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} = I_{(n+1)/2}$
$n \equiv_4 2$	$\begin{pmatrix} I_{(n-2)/2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} I_{(n-2)/2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$
$n \equiv_4 3$	$\begin{pmatrix} I_{(n-3)/2} & & & \\ & 0 & 1 & 0 \\ & 1 & 0 & 1 \\ & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} I_{(n-3)/2} & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} = I_{(n+1)/2}$

- $*$ For the $n \equiv_4 0$ case, optimal strategies end as $\boldsymbol{p} = (\dots, 0, p, p, 0)$ and $\boldsymbol{q} = (\dots, q, q, 0, 0)$, giving $p = q = V = 2/n$.
- $*$ For the $n \equiv_4 1$ case, optimal strategies end as $\boldsymbol{p} = (\dots, 0, p, p, p, 0)$ and $\boldsymbol{q} = (\dots, q, q, 0, 0, q)$, giving $p = q = V = 2/(n+1)$.
- $*$ For the $n \equiv_4 2$ case, optimal strategies end as $\boldsymbol{p} = (\dots, p, p)$ and $\boldsymbol{q} = (\dots, q, q)$, giving $p = q = V = 2/(n+2)$.
- $*$ Finally, for the $n \equiv_4 3$ case, optimal strategies end as $\boldsymbol{p} = (\dots, 0, p, p)$ and $\boldsymbol{q} = (\dots, q, q, 0)$, giving $p = q = V = 2/(n+1)$.

2.3.1. (a) There is a saddle point at $(2, 3)$, hence we determine immediately that the value of the game is 1.

(b) Any reasonable mathematical software package can be used to find that

$$A^{-1} = \begin{pmatrix} -2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -3 & 6 & -4 \end{pmatrix}.$$

Alternatively, to just show existence, we need only verify that the $\det(A) = 5 \neq 0$.

(c) Any $\boldsymbol{q} > 0$ such that $A\boldsymbol{q} \leq 1$ (that is, each component of $A\boldsymbol{q}$ is no more than $V = 1$) will do. One such example is $\boldsymbol{q} = (1/4, 1/2, 1/4)^T$.

(d) This can only be because player I does *not* have an optimal strategy giving positive weight to all rows. Indeed, since player II *does* have an optimal strategy giving all columns positive weight, then player I's optimal strategy $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = VA^{-T}\mathbf{1} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T.$$

This makes sense, since any deviation of this optimal strategy requires player I to "transfer" some weight from row 2 to either or both of rows 1 and 3. This can be seen to be suboptimal, as follows. Let p_1 and p_3 denote the probabilities that player I chooses rows 1 and 3, respectively. Then, if $p_1 \geq p_3$ and $p_1 > 0$, the expected payoff when player II chooses column 1 is

$$-2p_1 + (1 - p_1 - p_3) + 3p_3 = 1 - 2(p_1 - p_3) - p_1 < 1,$$

i.e., player I is no longer guaranteed an expected payoff of at least 1. Likewise, if $p_3 > p_1$, then the expected payoff when player II chooses column 2 is

$$2p_1 + (1 - p_1 - p_3) + 0p_3 = 1 - (p_3 - p_1) < 1.$$

It follows that it is optimal for $p_1 = p_3 = 0$, and player I should always choose row 2.

2.3.7. We can visualize these matrices by writing out the terms in a small upper-left block:

$$A_n = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ -1 & 3 & -3 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, B_n = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By Theorem 3.2 (provided that both $\boldsymbol{p}$ and $\boldsymbol{q}$ will be nonnegative), the value of the above game is

$$V = \left(\mathbf{1}^T A_n^{-1} \mathbf{1}\right)^{-1} = \left(\mathbf{1}^T B_n \mathbf{1}\right)^{-1}.$$

We are given that row i of B_n are the coefficients of the expansion of $(x+y)^{i-1}$, hence the sum of these coefficients is given by substituting $x = y = 1$, i.e., 2^{i-1}. Thus, $B_n\mathbf{1} = \left(1, 2, 4, \ldots, 2^{n-1}\right)^T$. Multiplying on the left by $\mathbf{1}^T$ sums the components of $B_n\mathbf{1}$, in the end giving $V = 1/(2^n - 1)$. The optimal strategy $\boldsymbol{q}$ for player II then follows immediately:

$$\boldsymbol{q} = VA_n^{-1}\mathbf{1} = VB_n\mathbf{1} = \frac{1}{2^n - 1}\left(1, 2, 4, \ldots, 2^{n-1}\right)^T.$$

The optimal strategy $\boldsymbol{p}$ for player I is similarly given by

$$\boldsymbol{p} = VA_n^{-1}\mathbf{1} = VB_n^T\mathbf{1}.$$

Evaluating $B_n^T\mathbf{1}$ amounts to summing down a column of B_n. In other words, the j^{th} component of $B_n^T\mathbf{1}$ is

$$\left(B_n^T\mathbf{1}\right)_j = \sum_{i=j}^{n}\binom{i-1}{j-1}.$$

Using equation (10) from `http://en.wikipedia.org/wiki/Binomial_coefficients` (or computing out the first few values by hand and proving by induction), we find that $\left(B_n^T\mathbf{1}\right)_j = \binom{n}{j}$, so

$$\boldsymbol{p} = \frac{1}{2^n - 1}\left(\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n} = 1\right)^T.$$

2.3.14. (a) With 3 posts among which to distribute his 4 units, Blotto has $\binom{4+3-1}{3-1} = 15$ pure strategies. Similarly, Kije, with 3 units, has $\binom{3+3-1}{3-1} = 10$ pure strategies. Hence, the full game matrix would consist of $15 \cdot 10 = 150$ entries. This can be significantly reduced by identifying invariant classes of strategies. The transformations under which this game is invariant simply consist of relabelings of the posts. To be specific, in the notation of the section, $\mathcal{G} = \{g_{123}, g_{132}, g_{213}, g_{231}, g_{312}, g_{321}\}$, where g_{ijk} represents the relabeling of posts $1, 2, 3$ to i, j, k, respectively. For example, g_{132} interchanges posts 2 and 3, leaving post 1 alone; and g_{123} is the identity permutation. Further, since relabeling the posts has precisely the same effect on the strategies of Blotto as on the strategies of Kije, $\overline{g_{ijk}} = g_{ijk}$, and so $\overline{\mathcal{G}} = \mathcal{G}$. As an example of how these transformations map the strategies of each player, consider g_{132}. Blotto's pure strategies get transformed like

$$g_{132}((4,0,0)) = (4,0,0); \quad g_{132}((3,1,0)) = (3,0,1); \quad g_{132}((3,0,1)) = (3,1,0); \quad \ldots;$$

that is, g_{132} effectively just interchanges the number of units Blotto assigns to posts 2 and 3. The effect on Kije's pure strategies is similar. After analyzing all 6 transformations, we find that Blotto has 4 invariant classes of strategies, while Kije has 3 (within an invariant class of strategies, we can get between any two pure strategies via some g_{ijk} transformations above). Blotto's invariant classes of strategies are

$$\begin{aligned}
(4,0,0)^* &= \{(4,0,0), (0,4,0), (0,0,4)\} \\
(3,1,0)^* &= \{(3,1,0), (3,0,1), (1,3,0), (1,0,3), (0,3,1), (0,1,3)\} \\
(2,2,0)^* &= \{(2,2,0), (2,0,2), (0,2,2)\} \\
(2,1,1)^* &= \{(2,1,1), (1,2,1), (1,1,2)\};
\end{aligned}$$

and Kije's invariant classes of strategies are

$$\begin{aligned}
(3,0,0)^* &= \{(3,0,0), (0,3,0), (0,0,3)\} \\
(2,1,0)^* &= \{(2,1,0), (2,0,1), (1,2,0), (1,0,2), (0,2,1), (0,1,2)\} \\
(1,1,1)^* &= \{(1,1,1)\}.
\end{aligned}$$

Since there must exist invariant optimal strategies for both players, we may assume that the pure strategies within each class are given equal probability. Stated another way, any probability we assign to an invariant class of strategies is distributed equally among its constituent pure strategies. This reduces the original 15×10 matrix down to a 4×3 matrix, whose entries are the expected payoffs when Blotto and Kije each choose (with equal probability) among their pure strategies within the particular invariant classes of strategies. Effectively, then, each entry in the 4×3 reduced matrix is simply an average of some (not necessarily contiguous) submatrix of the original 15×10 matrix, so we need only identify this submatrix for each pair of invariant classes of strategies for Blotto and Kije. For example, to compute the expected payoff when Blotto chooses $(4, 0, 0)^*$ and Kije chooses $(3, 0, 0)^*$, we take the average of the entries of the 3×3 submatrix

	$(3,0,0)$	$(0,3,0)$	$(0,0,3)$
$(4,0,0)$	4	0	0
$(0,4,0)$	0	4	0
$(0,0,4)$	0	0	4

.

Thus, the $(1, 1)$ entry (which we assume corresponds to Blotto choosing $(4, 0, 0)^*$ and Kije choosing $(3, 0, 0)^*$) in the reduced 4×3 game matrix is $12/9 = 4/3$. Notice that any particular row (column) of the above submatrix is a permutation of the other rows (columns), which is a general property of all such submatrices. This is because any transformation in $\mathcal{G}$ just permutes the pure strategies within an invariant class of strategies, which has the effect of just permuting the corresponding payoffs when going from one row to another (or from one column to another) in one of these submatrices. The remaining entries to the reduced 4×3 matrix are computed using the payoffs below (where several rows or columns from different submatrices have been concatenated into the same payoff submatrices to save space):

	$(3,0,0)$	$(0,3,0)$	$(0,0,3)$	$(1,1,1)$
$(4,0,0)$	4	0	0	0
$(3,1,0)$	1	-1	1	1
$(2,2,0)$	-2	-2	1	3
$(2,1,1)$	-1	0	0	2

	$(2,1,0)$	$(2,0,1)$	$(1,2,0)$	$(1,0,2)$	$(0,2,1)$	$(0,1,2)$
$(4,0,0)$	2					
$(0,4,0)$	1					
$(0,0,4)$	-1					
$(3,1,0)$	3	3	0	2	-2	0
$(2,2,0)$	2					
$(2,0,2)$	0					
$(0,2,2)$	2					
$(2,1,1)$	1					
$(1,2,1)$	1					
$(1,1,2)$	-1					

The resulting reduced 4×3 game matrix is thus

	$(3,0,0)^*$	$(2,1,0)^*$	$(1,1,1)^*$
$(4,0,0)^*$	$4/3$	$2/3$	0
$(3,1,0)^*$	$1/3$	1	1
$(2,2,0)^*$	-1	$4/3$	3
$(2,1,1)^*$	$-1/3$	$1/3$	2

We now look for dominated rows and columns. Beginning with rows, we see that the column maxima occur in rows 1 and 3, so rows 2 and 4 are candidates for domination. Indeed, row 4 is less than 1/3 of row 1 plus 2/3 of row 3, giving the reduced 3×3 matrix

$$\begin{pmatrix} 4/3 & 2/3 & 0 \\ 1/3 & 1 & 1 \\ -1 & 4/3 & 3 \end{pmatrix}.$$

No further row domination is possible, so we now look for dominated columns. Row minima occur in columns 1 and 3, meaning that only column 2 is a candidate for domination. Indeed, a probability combination of 1/2 each of columns 1 and 3 dominates column 2, giving the reduced 3×2 matrix

$$\begin{pmatrix} 4/3 & 0 \\ 1/3 & 1 \\ -1 & 3 \end{pmatrix}.$$

Now, we see that 1/3 of row 1 plus 2/3 of row 3 dominates row 2, finally reducing our game to the 2×2 matrix $\left(\begin{smallmatrix} 4/3 & 0 \\ -1 & 3 \end{smallmatrix}\right)$. This solves easily to give

$$V = \frac{4/3 \cdot 3 - 0 \cdot (-1)}{4/3 - 0 + 3 - (-1)} = \frac{3}{4}; \quad p = \frac{3 - (-1)}{4/3 - 0 + 3 - (-1)} = \frac{3}{4}; \quad q = \frac{3 - 0}{4/3 - 0 + 3 - (-1)} = \frac{9}{16}.$$

Extrapolating back to the reduced 4×3 game, this gives strategies for Blotto and Kije of $(3/4, 0, 1/4, 0)$ and $(9/16, 0, 7/16)$, respectively. The value is still $V = 3/4$. It is a simple exercise to check that these satisfy the principle of invariance, so are correct (assuming the reduction to the 4×3 matrix is correct). We interpret the strategies in the context of the original game as follows:

* An optimal strategy for Blotto consists of sending all 4 units to post i with probability 1/4, for $i = 1, 2, 3$; and sending 2 units each to posts i and j with probability 1/4, for $(i, j) = (1, 2), (1, 3), (2, 3)$.
* An optimal strategy for Kije consists of sending all 3 units to post i with probability 3/16, for $i = 1, 2, 3$; and sending 1 unit each to all 3 posts with probability 7/16.

(b)

2.4.2. (a) We need only observe that $B = 2A + 5$, hence $\text{value}(B) = 2\,\text{value}(A) + 5 = 5$. Since the game represented by B is just a "change of location and scale" from the game represented by A, the optimal strategies for both games are the same. In particular, an optimal strategy for player I is $(6/11, 3/11, 2/11)$.

(b) As mentioned above, a strategy for player II is optimal for A if and only if it is optimal for B, so we may choose to use either matrix to find an optimal strategy for player II. Since player I has an optimal strategy giving positive weight to each of his rows, we know, by the principle of indifference, that any optimal strategy for player II must be an equalizing strategy, i.e., it must give the same expected payoff no matter which row player I chooses. We *could* use Theorem 3.2 on page II - 20 to compute $\boldsymbol{q}$ using the game matrix B (we can't use A since it is singular), but it turns out to be easiest to just write out the system of equations that $\boldsymbol{q}$ must satisfy with respect to the game matrix A, and solving. This quickly gives $\boldsymbol{q} = (1/3, 1/3, 1/3)$ as an optimal strategy for player II.

2.4.3. (a) Let $\boldsymbol{q} \in Y^*$. Then, since $A(i, j) = 1$ for $j < i$; $A(i, i) = 0$; and $A(i, j) = -1$ for $j > i$; we have

$$\sum_{j=1}^{\infty} A(i, j) q_j = \sum_{j=1}^{i-1} q_j - \sum_{j=i+1}^{\infty} q_j,$$

and hence

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} A(i, j) q_j = \lim_{i\to\infty} \sum_{j=1}^{i-1} q_j - \lim_{i\to\infty} \sum_{j=i+1}^{\infty} q_j = 1 - 0 = 1,$$

using the fact that $\sum_j q_j = 1$. Further, since each $q_j \geq 0$, we have the inequality

$$\sum_{j=1}^{\infty} A(i,j)q_j \leq \sum_{j=1}^{i-1} q_j \leq \sum_{j=1}^{\infty} q_j = 1$$

for all $i \geq 1$, so, indeed,

$$\sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} A(i,j)q_j = \lim_{i \to \infty} \sum_{j=1}^{\infty} A(i,j)q_j = 1.$$

(b) This claim follows straight from (a) and the definition of $\overline{V}$ (replacing max's and min's with sup's and inf's, respectively):

$$\overline{V} = \inf_{\boldsymbol{q} \in Y^*} \sup_{1 \leq i < \infty} (A\boldsymbol{q})_i = \inf_{\boldsymbol{q} \in Y^*} 1 = 1.$$

(c) The game is completely symmetric, so we should have $\underline{V} = -\overline{V} = -1$. More explicitly, since $A^T = -A$ and $X^* = Y^*$,

$$\begin{aligned}
\underline{V} &= \sup_{\boldsymbol{p} \in X^*} \inf_{1 \leq j < \infty} \left(\boldsymbol{p}^T A\right)_j \\
&= \sup_{\boldsymbol{p} \in X^*} \inf_{1 \leq i < \infty} \left(A^T \boldsymbol{p}\right)_i \\
&= \sup_{\boldsymbol{p} \in X^*} \inf_{1 \leq i < \infty} -(A\boldsymbol{p})_i \\
&= -\inf_{\boldsymbol{p} \in X^*} \sup_{1 \leq i < \infty} (A\boldsymbol{p})_i \\
&= -\overline{V}.
\end{aligned}$$

(d) By definition, a minimax strategy for player I is one which achieves the maximum (or, in this case, the supremum) in the expression for $\underline{V}$. But since

$$\inf_{1 \leq j < \infty} \left(\boldsymbol{p}^T A\right)_j = -1$$

for *any* $\boldsymbol{p}$, it follows that *any* strategy is a minimax strategy for player I. Said another way, any strategy for player I is guarantees a loss of at most 1 to player II no matter how player II plays. This should be obvious from inspection of the game matrix A: all entries are at least -1, so no outcome can result in player I paying more than 1 to player II (and vice versa)!

2.5.9. (a) Here is a rough sketch of the game tree.

```
                   I
                  / \
            H (A) /   \ T (B)
           ______/     \______
          /                 \
         N                   N
       _/ \_               _/ \_
   1/2 /    \ 1/2      1/3 /    \ 2/3
    H /      \ T        H /      \ T
     /        \          /        \
    II        II'       II'       II
    /\        /\        /\        /\
  A/  \B   A'/  \B'  A'/  \B'   A/  \B
  /    \    /    \    /    \    /    \
-1     2  -1     0   0    -1   2    -1
```

The node labelled `I` comprises an information set for player I, while the pairs of nodes labelled `II` and `II'`, respectively, comprise information sets for player II.

(b) We first identify the pure strategies for each player. Recall that a pure strategy for a given player consists of a tuple of branch choices, the size of the tuple being the number of information sets for that player, and each tuple element corresponding to a branch choice out of the corresponding information set. Thus, the pure strategy space for player I is $X = \{H, T\}$, and the pure strategy space for player II is $Y = \{(A, A'), (A, B'), (B, A'), (B, B')\}$. We compute the expected payoff (to player I) when each player uses a pure strategy to get the following 2×4 game matrix:

	(A, A')	(A, B')	(B, A')	(B, B')
H	-1	$-1/2$	$1/2$	1
T	$4/3$	1	$-2/3$	-1

(c) This matrix can be reduced via column domination: column 2 is dominated by taking 3/4 of column 1 plus 1/4 of column 4. This gives the 2×3 reduced matrix $\begin{pmatrix} -1 & 1/2 & 1 \\ 4/3 & -2/3 & -1 \end{pmatrix}$. If player I chooses row 1 in this reduced system with probability p, we can graph the expected payoffs as functions of p when player II chooses each of columns 1, 2, and 3 in this reduced system. The optimal choice of p for player I is that which maximizes the lower envelope, which occurs at the intersection of columns 1 and 2. Thus, the we arrive at the reduced 2×2 matrix $\begin{pmatrix} -1 & 1/2 \\ 4/3 & -2/3 \end{pmatrix}$, which easily solves to give $V = 0$, $p = 4/7$, and $q = 1/3$. Extrapolating back to the original strategic game matrix, this gives optimal strategies for players I and II of $\boldsymbol{p} = (4/7, 3/7)$ and $\boldsymbol{q} = (1/3, 0, 2/3, 0)$, respectively (the value of the game remains $V = 0$). In the context of the original game, this means that, in the beginning, player I should choose H with probability 4/7 and T with probability 3/7. If player II is informed that player I is wrong, she should always guess coin B. On the other hand, if player II is informed that player I is right, she should guess coin A 1/3 of the time and coin B 2/3 of the time.

3.1.3. (a) The Kuhn tree could look as follows, with payoffs in units of \$100. Note that since contestants I and II move simultaneously, either could be regarded as moving "first". Further, we have taken the liberty to combine successive nature moves into one node only to give a more compact representation of the tree.

```
                    I
       ------------/ \------------
    P /                           \ G
     II                            II
  P' /\ G'               ---------/\---------
    /  \             P' /                   \ G'
(4,9)   N              N                     N
   1/2 / \ 1/2    1/2 / \ 1/2         ---------/ \---------
  (4,11) (8,3)   (10,5) (0,4)   1/4 /    1/4 /    \ 1/4    \ 1/4
                                 (6,11) (10,3) (0,11)  (0,7)
```

The vertices at which contestant II moves comprise a single information set.

(b) The pure strategies for each player are either pass (P / P') or gamble (G / G'). The bimatrix for the strategic form of the game thus looks like

	P'	G'
P	$(4, 9)$	$(6, 7)$
G	$(5, 7)$	$(4, 8)$

.

(c) The safety levels are numerically equal to the values of the individual payoff matrices for each contestant:

$$v_I = \text{value}(A), \quad v_{II} = \text{value}(B^T),$$

where

$$A = \begin{pmatrix} 4 & 6 \\ 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & 7 \\ 7 & 8 \end{pmatrix}.$$

These are easily computed to be

$$v_I = \frac{16 - 30}{4 - 6 + 4 - 5} = \frac{14}{3}, \quad v_{II} = \frac{72 - 49}{9 - 7 + 8 - 7} = \frac{23}{3}.$$

The corresponding maxmin strategies are $\boldsymbol{p} = (1/3, 2/3)$ and $\boldsymbol{q} = (1/3, 2/3)$. Thus, to maximize their expected payoff independent of what the other contestant does, both contestants should pass with probability 1/3 and gamble with probability 2/3. Contestant I would be guaranteed an expected payoff of at least \$1400/3, while contestant II would be guaranteed an expected payoff of at least \$2300/3.

3.2.6. (a) Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

be the individual payoff matrices for each player. Notice that A has a saddle point of 0, while the columns 1 and 3 of B are equivalent (and hence can be combined). Thus,

$$v_I = \text{value}(A) = 0, \quad v_{II} = \text{value}(B^T) = \frac{2}{3}.$$

Any strategy for player I is a maxmin strategy, while any strategy giving weight 1/3 to column 2 is a maxmin strategy for player II.

(b) The only pure strategic equilibrium occurs at row 2, column 1, and has payoff $(0, 1)$.

(c) An equalizing strategy for player I equalizes the expected payoffs of each column in player II's payoff matrix, B. Thus, $\boldsymbol{p}$ is an equalizing strategy for player I if

$$0p_1 + 1p_2 = 2p_1 + 0p_2 = 0p_1 + 1p_2,$$

which, though overdetermined, has the solution $\boldsymbol{p} = (1/3, 2/3)$. Similarly, an equalizing strategy $\boldsymbol{q}$ for player II satisfies

$$0q_1 + 1q_2 + 2q_3 = 0q_1 + 2q_2 + 0q_3,$$

so $\boldsymbol{q} = (1 - 3x, 2x, x)$ for any $x \in [0, 1/3]$. The payoff of this equalizing strategic equilibrium is $(4x, 2/3)$.